From Physics to Number theory via Noncommutative Geometry

Alain Connes and Matilde Marcolli

Contents

Introduction

e volta nostra poppa nel mattino,
de' remi facemmo ali al folle volo
—— Dante, Inf. XXVI 124-125

Several recent results reveal a surprising connection between modular forms and noncommutative geometry. The first occurrence came from the classification of noncommutative three spheres, [C–DuboisViolette-I] [C–DuboisViolette-II]. Hard computations with the noncommutative analog of the Jacobian involving the ninth power of the Dedekind eta function were necessary in order to analyze the relation between such spheres and noncommutative nilmanifolds. Another occurrence can be seen in the computation of the explicit cyclic cohomology Chern character of a spectral triple on $SU_q(2)$ [C–02]. Another surprise came recently from a remarkable action of the Hopf algebra of transverse geometry of foliations of codimension one on the space of lattices modulo Hecke correspondences, described in the framework of noncommutative geometry, using a modular Hecke algebra obtained as the cross product of modular forms by the action of Hecke correspondences [C–Moscovici-I] [C–Moscovici-II]. This action determines a differentiable structure on this noncommutative space, related to the Rankin–Cohen brackets of modular forms, and shows their compatibility with Hecke operators. Another instance where properties of modular forms can be recast in the context of noncommutative geometry can be found in the theory of modular symbols and Mellin transforms of cusp forms of weight two, which can be recovered from the geometry of the moduli space of Morita equivalence classes of noncommutative tori viewed as boundary of the modular curve [Manin–M].

In this paper we show that the theory of modular Hecke algebras, the spectral realization of zeros of L-functions, and the arithmetic properties of KMS states in quantum statistical mechanics combine into a unique general picture based on the noncommutative geometry of the space of commensurability classes of $\mathbb{Q}$-lattices.

An n-dimensional $\mathbb{Q}$-lattice consists of an ordinary lattice Λ in $\mathbb{R}^n$ and a homomorphism

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda.$$

Two such $\mathbb{Q}$-lattices are *commensurable* if and only if the corresponding lattices are commensurable and the maps agree modulo the sum of the lattices.

The description of the spaces of commensurability classes of $\mathbb{Q}$-lattices via non-commutative geometry yields two quantum systems related by a duality. The first system is of quantum statistical mechanical nature, with the algebra of coordinates parameterizing commensurability classes of $\mathbb{Q}$-lattices modulo scaling and with a time evolution with eigenvalues given by the index of pairs of commensurable $\mathbb{Q}$-lattices. There is a symmetry group acting on the system, in general by *endomorphisms*. It is this symmetry that is spontaneously broken at low temperatures, where the system exhibits distinct phases parameterized by arithmetic data. We completely analyze the phase transition with spontaneous symmetry breaking in the two-dimensional case, where a new phenomenon appears, namely that there is a second critical temperature, beyond which no equilibrium state survives.

In the "dual system", which corresponds just to commensurability of $\mathbb{Q}$-lattices, the scaling group is acting. In physics language, what emerges is that the zeros of zeta appear as an absorption spectrum of the scaling action in the L^2 space of the space of commensurability classes of $\mathbb{Q}$-lattices as in [C–99]. While the zeros of zeta and L-functions appear at the critical temperature, the analysis of the low temperature equilibrium states concentrates on the subspace

$$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$$

of *invertible* $\mathbb{Q}$-lattices, which as is well known plays a central role in the theory of automorphic forms.

While, at first sight, at least in the 1-dimensional case, it would seem easy to classify commensurability classes of $\mathbb{Q}$-lattices, we shall see that ordinary geometric tools fail because of the ergodic nature of the equivalence relation.

Such quotients are fundamentally of "quantum nature", in that, even though they are sets in the ordinary sense, it is impossible to distinguish points by any finite (or countable) collection of invariants. Noncommutative geometry is specifically designed to handle such quantum spaces by encoding them by algebras of non-commuting coordinates and extending the techniques of ordinary geometry using the tools of functional analysis, noncommutative algebra, and quantum physics.

Direct attempts to define function spaces for such quotients lead to invariants that are of a cohomological nature. For instance, let the fundamental group Γ of a Riemann surface act on the boundary $\mathbb{P}^1(\mathbb{R})$ of its universal cover identified with the Poincaré disk. The space

$$L^\infty(\Gamma\backslash\mathbb{P}^1(\mathbb{R})) := L^\infty(\mathbb{P}^1(\mathbb{R}))^\Gamma$$

is in natural correspondence with global sections of the sheaf of (real parts of) holomorphic functions on the Riemann surface, as boundary values. More generally, the cyclic cohomology of the noncommutative algebra of coordinates on such quotients is obtained by applying derived functors to these naive functorial definition of function spaces.

In the 1-dimensional case, the states at zero temperature are related to the Kronecker–Weber construction of the maximal abelian extension $\mathbb{Q}^{ab}$. In fact, in this case the quantum statistical mechanical system is the one constructed in [Bost–C], which has underlying geometric space X_1 parameterizing commensurability classes of 1-dimensional $\mathbb{Q}$-lattices modulo scaling by $\mathbb{R}_+^*$. The corresponding algebra of coordinates is a Hecke algebra for an almost normal pair of solvable groups. The regular representation is of type III_1 and determines the time evolution of the system, which has the set of $\log(p)$, p a prime number, as set of basic frequencies. The system has an action of the idèles class group modulo the connected component of identity as a group of symmetries. This induces a Galois action on the ground states of the system at zero temperature. When raising the temperature the system has a phase transition, with a unique equilibrium state above the critical temperature. The Riemann zeta function appears as the partition function of the system, as in [Julia].

Each equivalence class of $\mathbb{Q}$-lattices determines an irreducible covariant representation, where the Hamiltonian is implemented by minus the log of the covolume. For a general class, this is not bounded below. It is so, however, in the case of equivalence classes of *invertible* $\mathbb{Q}$-lattices, *i.e.* where the labelling of torsion points is one to one. These classes then define positive energy representations and corresponding KMS states for all temperatures below critical.

In the 2-dimensional case, as the temperature lowers, the system settles down on these invertible $\mathbb{Q}$-lattices, so that the zero temperature space is commutative and is given by the Shimura variety

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*.$$

The action of the symmetry group, which in this case is nonabelian and isomorphic to $\mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)$, is more subtle due to the presence of inner automorphisms and the necessary use of the formalism of superselection sectors. Moreover, its effect on the zero temperature states is not obtained directly but is induced by the action at non-zero temperature, which involves the full noncommutative system. The quotient $\mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f)\times \mathrm{GL}_2(\mathbb{R}))/\mathbb{C}^*$ and the space of 2–dimensional $\mathbb{Q}$-lattices modulo commensurability and scaling are the same, hence the corresponding algebras are Morita equivalent. However, it is preferable to work with the second description, since, by taking the classical quotient by the action of the subgroup $\mathrm{SL}_2(\mathbb{Z})$, it reduces the group part in the cross product to the classical Hecke algebra.

The GL_2 system has an arithmetic structure provided by a *rational* subalgebra, given by a natural condition on the coefficients of the q-series. We show that it is a Hecke algebra of modular functions, closely related to the modular Hecke algebra of [C–Moscovici-I], [C–Moscovici-II]. The symmetry group acts on the values of ground states on this rational subalgebra as the automorphism group of the modular field.

Evaluation of a generic ground state φ of the system on the rational subalgebra generates an embedded copy of the modular field in $\mathbb{C}$ and there exists a unique

isomorphism of the symmetry group of the system with the Galois group of the embedded modular field, which intertwines the Galois action on the image with the symmetries of the system,

$$\theta(\sigma) \circ \varphi = \varphi \circ \sigma.$$

The relation between this GL_2 system and class field theory is being investigated in ongoing work [C–M–Ramachandran].

The arithmetic structure is inherited by the dual of the GL_2 system and enriches the structure of the noncommutative space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices to that of a "noncommutative arithmetic variety". The relation between this dual system and the spectral realization of zeros of L-functions is the central topic of Chapter 2.

The dual of the GL_1 system, under the duality obtained by taking the cross product by the time evolution, corresponds to the space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices, not considered up to scaling. This corresponds geometrically to the total space $\mathcal{L}$ of a principal $\mathbb{R}_+^*$ bundle over the base X_1, and determines a natural scaling action of $\mathbb{R}_+^*$. The space $\mathcal{L}$ is described by the quotient

$$\mathcal{L} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot},$$

where $\mathbb{A}^{\cdot}$ denotes the set of adèles with nonzero archimedean component. The corresponding algebra of coordinates is Morita equivalent to $C(X_1) \rtimes_{\sigma_t} \mathbb{R}$.

Any approach to a spectral realization of the zeros of zeta through the quantization of a classical dynamical system faces the problem of obtaining the leading term in the Riemann counting function for the number of zeros of imaginary part less than E as a volume in phase space. The solution [C–99] of this issue is achieved in a remarkably simple way, by the scaling action of $\mathbb{R}_+^*$ on the phase space of the real line $\mathbb{R}$, and will be the point of departure for the second part of the paper.

In particular, this shows that the space $\mathcal{L}$ requires a further compactification at the archimedean place, obtained by replacing the quotient $\mathcal{L} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot}$ by $\overline{\mathcal{L}} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}$ *i.e.* dropping the non vanishing of the archimedean component. This compactification has an analog for the GL_2 case, given by the noncommutative boundary of modular curves considered in [Manin–M], which corresponds to replacing $\mathrm{GL}_2(\mathbb{R})$ by $M_2(\mathbb{R})$ at the archimedean place, and is related to class field theory for real quadratic fields through Manin's real multiplication program.

The space $\overline{\mathcal{L}}$ appears as the configuration space for a quantum field theory, where the degrees of freedom are parameterized by prime numbers, including infinity. When only finitely many degrees of freedom are considered, and in particular only the place at infinity, the semiclassical approximation exhibits the main terms in the asymptotic formula for the number of zeros of the Riemann zeta function.

The zeros of zeta appear as an absorption spectrum, namely as lacunae in a continuous spectrum, where the width of the absorption lines depends on

the presence of a cutoff. The full idèles class group appears as symmetries of the system and L-functions with Grössencharakter replace the Riemann zeta function in nontrivial sectors.

From the point of view of quantum field theory, the field configurations are given by adèles, whose space $\mathbb{A}$ is then divided by the action of the gauge group $\mathrm{GL}_1(\mathbb{Q})$. As mentioned above, the quotient space is essentially the same as the space $\mathcal{L}$ of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices. The $\log(p)$ appear as periods of the orbits of the scaling action. The Lefschetz formula for the scaling action recovers the Riemann–Weil explicit formula as a semi-classical approximation. The exact quantum calculation for finitely many degrees of freedom confirms this result. The difficulty in extending this calculation to the global case lies in the quantum field theoretic problem of passing to infinitely many degrees of freedom.

The main features of the dual systems in the GL_1 case are summarized in the following table:

Quantum statistical mechanics	Quantum field theory
Commensurability classes of $\mathbb{Q}$-lattices modulo scaling	Commensurability classes of $\mathbb{Q}$-lattices
$A = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$	$A \rtimes_{\sigma_t} \mathbb{R}$
Time evolution σ_t	Energy scaling $U(\lambda)$, $\lambda \in \mathbb{R}_+^*$
$\{\log p\}$ as frequencies	$\{\log p\}$ as periods of orbits
Arithmetic rescaling μ_n	Renormalization group flow $\mu\partial_\mu$
Symmetry group $\hat{\mathbb{Z}}^*$ as Galois action on $T=0$ states	Idèles class group as gauge group
System at zero temperature	$\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})$
System at critical temperature (Riemann's ζ as partition function)	Spectral realization (Zeros of ζ as absorption spectrum)
Type III_1	Type II_∞

There is a similar duality (and table) in the GL_2 case, where part of the picture remains to be clarified. The relation with the modular Hecke algebra of [C–Moscovici-I], [C–Moscovici-II] is more natural in the dual system where modular forms with non-zero weight are naturally present.

The dual system $\mathcal{L}$ can be interpreted physically as a "universal scaling system", since it exhibits the continuous renormalization group flow and its relation with the discrete scaling by powers of primes. For the primes two and three, this discrete scaling manifests itself in acoustic systems, as is well known in western classical music, where the two scalings correspond, respectively, to passing to the octave (frequency ratio of 2) and transposition (the perfect fifth is the frequency ratio 3/2), with the approximate value $\log(3)/\log(2) \sim 19/12$ responsible for the difference between the "circulating temperament" of the Well Tempered Clavier and the "equal temperament" of XIX century music. It is precisely the irrationality of $\log(3)/\log(2)$ which is responsible for the noncommutative nature of the quotient corresponding to the three places $\{2, 3, \infty\}$.

The key role of the continuous renormalization group flow as a symmetry of the dual system $\mathcal{L}$ and its similarity with a Galois group at the archimedean place brings us to the last part of this paper. In Chapter 3, we analyze the quantum statistical mechanics of $\mathbb{Q}$-lattices at critical temperature.

The fact that the KMS state at critical temperature can be expressed as a noncommutative residue (Dixmier trace) shows that the system at critical temperature should be analyzed with tools from quantum field theory and renormalization.

The mathematical theory of renormalization in QFT developed in [C–Kreimer-I] [C–Kreimer-II] shows in geometric terms that the procedure of perturbative renormalization can be described as the Birkhoff decomposition

$$g_{\mathrm{eff}}(\varepsilon) = g_{\mathrm{eff}_+}(\varepsilon)\,(g_{\mathrm{eff}_-}(\varepsilon))^{-1}$$

on the projective line of complexified dimensions ε of the loop

$$g_{\mathrm{eff}}(\varepsilon) \in G = \text{ formal diffeomorphisms of } \mathbb{C}$$

given by the unrenormalized effective coupling constant. The g_{eff_-} side of the Birkhoff decomposition yields the counterterms and the g_{eff_+} side evaluated at the critical dimension gives the renormalized value of the effective coupling. This explicit knowledge of the counterterms suffices to determine the full renormalized theory.

The principal G-bundle on $\mathbb{P}^1(\mathbb{C})$, with trivialization given by the Birkhoff decomposition, has a flat connection with regular singularities, coming from a Riemann–Hilbert problem determined by the representation datum given by the β-function of renormalization, viewed as the logarithm of the monodromy around the singular dimension. The problem of incorporating nonperturbative effects leads to a more sophisticated Riemann–Hilbert problem in terms of a representation of the wild fundamental group of Martinet–Ramis, related to

applying Borel summation techniques to the unrenormalized effective coupling constant.

In the perturbative theory the renormalization group appears as a natural 1-parameter subgroup of the group of "diffeographisms" which governs the ambiguity in the choice of the physical solution. In fact, the renormalization group, the wild fundamental group and the connected component of identity in the idèles class group are all incarnations of a still mysterious Galois theory at the archimedean place.

It was shown in [C–00] that the classification of approximately finite factors provides a nontrivial Brauer theory for central simple algebras over $\mathbb{C}$, and an archimedean analog of the module of central simple algebras over nonarchimedean fields. The relation of Brauer theory to the Galois group is via the construction of central simple algebras as cross products of a field by a group of automorphisms. It remained for a long time an elusive point to obtain in a natural manner factors as cross product by a group of automorphisms of a *field*, which is a transcendental extension of $\mathbb{C}$. This was achieved in [C–DuboisViolette-II] for type II_1, via the cross product of the field of elliptic functions by an automorphism given by translation on the elliptic curve. The results on the GL_2 system give an analogous construction for type III_1 factors using the modular field.

The physical reason for considering such field extensions of $\mathbb{C}$ lies in the fact that the coupling constants g of the fundamental interactions (electromagnetic, weak and strong) are not really constants but depend on the energy scale μ and are therefore functions $g(\mu)$. Thus, high energy physics implicitly extends the "field of constants", passing from the field of scalars $\mathbb{C}$ to a field of functions containing all the $g(\mu)$. On this field, the renormalization group provides the corresponding theory of ambiguity, and acquires an interpretation as the missing Galois group at archimedean places.

The structure of the paper is organized as follows.

- The first part is dedicated to the quantum statistical mechanical system of $\mathbb{Q}$-lattices, in the cases of dimension one and two, and its behavior at zero temperature.
- The second part deals with the GL_1 system at critical temperature and its dual system, and their relation to the spectral realization of the zeros of zeta of [C–99].
- The last part is dedicated to the system at critical temperature, the theory of renormalization of [C–Kreimer-I], [C–Kreimer-II], the Riemann-Hilbert problem and the missing Galois theory at the archimedean place.

Bibliography

[Bost–C] J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) Vol.1 (1995) N.3, 411–457.

[C–02] A. Connes, *Cyclic cohomology, Quantum group Symmetries and the Local Index Formula for $SU_q(2)$*.(2002), Math QA/0209142.

[C–00] A. Connes, *Noncommutative Geometry and the Riemann Zeta Function*, Mathematics: Frontiers and perspectives, IMU 2000 volume.

[C–99] A. Connes, *Trace formula in Noncommutative Geometry and the zeros of the Riemann zeta function*. Selecta Mathematica. (New Series) Vol.5 (1999) 29–106.

[C–DuboisViolette-I] A. Connes, M. Dubois-Violette, *Noncommutative finite-dimensional manifolds. I. spherical manifolds and related examples*, Comm. Math. Phys. Vol.230 (2002) N.3, 539–579.

[C–DuboisViolette-II] A. Connes, M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres*, Lett. Math. Phys., Vol.66 (2003) N.1-2, 91–121.

[C–Kreimer-I] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*. Comm. Math. Phys. 210 (2000), N.1, 249–273.

[C–Kreimer-II] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. II. The β-function, diffeomorphisms and the renormalization group*. Comm. Math. Phys. 216 (2001), N.1, 215–241.

[C–M–Ramachandran] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, in preparation.

[C–Moscovici-I] A. Connes, H. Moscovici, *Modular Hecke algebras and their Hopf symmetry*, Moscow Math. Journal, Vol.4 (2004) N.1, 67–109.

[C–Moscovici-II] A. Connes, H. Moscovici, *Rankin-Cohen Brackets and the Hopf Algebra of Transverse Geometry*, Moscow Math. Journal, Vol.4 (2004) N.1, 111–130.

[Julia] B. Julia, *Statistical theory of numbers*, in Number Theory and Physics, J.-M. Luck, P. Moussa and M. Waldschmidt (Eds.), Springer Verlag, Berlin, 1990.

[Manin–M] Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and noncommutative geometry*, Selecta Mathematica (New Series) Vol.8 (2002) N.3, 475–520.

Chapter 1

Quantum Statistical Mechanics of $\mathbb{Q}$-Lattices

1.1 Introduction

In this chapter we shall start by giving a geometric interpretation in terms of the space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices of the quantum statistical dynamical system (BC [5]). This system exhibits the relation between the phenomenon of spontaneous symmetry breaking and number theory. Its dual system obtained by taking the cross product by the time evolution is basic in the spectral interpretation of zeros of zeta.

Since $\mathbb{Q}$-lattices and commensurability continue to make sense in dimension n, we shall obtain an analogous system in higher dimension and in particular we derive a complete picture of the system in dimension $n = 2$. This shows two distinct phase transitions with arithmetic spontaneous symmetry breaking.

In the initial model of BC ([5]) the partition function is the Riemann zeta function. Equilibrium states are characterized by the KMS-condition. While at large temperature there is only one equilibrium state, when the temperature gets smaller than the critical temperature, the equilibrium states are no longer unique but fall in distinct phases parameterized by number theoretic data. The pure phases are parameterized by the various embeddings of the cyclotomic field $\mathbb{Q}^{ab}$ in $\mathbb{C}$.

The physical observables of the BC system form a C^*-algebra endowed with a natural time evolution σ_t. This algebra is interpreted here as the algebra of noncommuting coordinates on the space of commensurability classes of 1-dimensional $\mathbb{Q}$-lattices up to scaling by $\mathbb{R}_+^*$.

What is remarkable about the ground states of this system is that, when evaluated on the rational observables of the system, they only affect values that are algebraic numbers. These span the maximal abelian extension of $\mathbb{Q}$. Moreover, the class field theory isomorphism intertwines the two actions of the idèles class

group, as symmetry group of the system, and of the Galois group, as permutations of the expectation values of the rational observables. That the latter action preserves positivity is a rare property of states. We abstract this property as a definition of "fabulous[1] states", in the more general context of arbitrary number fields and review recent developments in the direction of extending this result to other number fields.

We present a new approach, based on the construction of an analog of the BC system in the GL_2 case. Its relation to the complex multiplication case of the Hilbert 12th problem will be discussed specifically in ongoing work of the two authors with N. Ramachandran [13].

The C^*-algebra of observables in the GL_2-system describes the non-commutative space of commensurability classes of $\mathbb{Q}$-lattices in $\mathbb{C}$ up to scaling by $\mathbb{C}^*$.

A $\mathbb{Q}$-lattice in $\mathbb{C}$ is a pair (Λ, ϕ) where $\Lambda \subset \mathbb{C}$ is a lattice while

$$\phi : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

is a homomorphism of abelian groups (not necessarily invertible). Two $\mathbb{Q}$-lattices (Λ_j, ϕ_j) are commensurable iff the lattices Λ_j are commensurable (*i.e.* $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps ϕ_j are equal modulo $\Lambda_1 + \Lambda_2$. The time evolution corresponds to the ratio of covolumes of pairs of commensurable $\mathbb{Q}$-lattices. The group

$$S = \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$$

quotient of the finite adèlic group of GL_2 by the multiplicative group $\mathbb{Q}^*$ acts as symmetries of the system, and the action is implemented by endomorphisms, as in the theory of superselection sectors of Doplicher-Haag-Roberts ([16]).

It is this symmetry which is spontaneously broken below the critical temperature $T = \frac{1}{2}$. The partition function of the GL_2 system is $\zeta(\beta)\zeta(\beta - 1)$, for $\beta = 1/T$, and the system exhibits three distinct phases, with two phase transitions at $T = \frac{1}{2}$ and at $T = 1$. At low temperatures ($T < \frac{1}{2}$) the pure phases are parameterized by the set

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$$

of classes of invertible $\mathbb{Q}$-lattices (up to scaling). The equilibrium states of the "crystalline phase" merge as $T \to 1/2$ from below, as the system passes to a "liquid phase", while at higher temperatures ($T \geq 1$) there are no KMS states.

The subalgebra of *rational* observables turns out to be intimately related to the modular Hecke algebra introduced in (Connes-Moscovici [10]) where its surprising relation with transverse geometry of foliations is analyzed. We show that the KMS states at zero temperature when evaluated on the rational observables generate a specialization of the modular function field F. Moreover, as in the BC system the state intertwines the two actions of the group S, as symmetry

[1]This terminology is inspired from John Conway's talk on "fabulous" groups.

group of the system, and as permutations of the expectation values of the rational observables by the Galois group of the modular field, identified with S by Shimura's theorem ([51]).

We shall first explain the general framework of quantum statistical mechanics, in terms of C^*-algebras and KMS states. Noncommutative algebras concretely represented in Hilbert space inherit a canonical time evolution, which allows for phenomena of phase transition and spontaneous symmetry breaking for KMS states at different temperatures.

There are a number of important nuances between the abelian BC case and the higher dimensional non-abelian cases. For instance, in the abelian case, the subfield of $\mathbb{C}$ generated by the image of the rational subalgebra under an extremal KMS_∞ state does not depend on the choice of the state and the intertwining between the symmetry and the Galois actions is also independent of the state. This no longer holds in the non-abelian case, because of the presence of inner automorphisms of the symmetry group S.

Moreover, in the GL_2 case, the action of S on the extremal KMS_∞ states is not transitive, and the corresponding invariant of the orbit of a state φ under S is the subfield $F_\varphi \subset \mathbb{C}$, which is the specialization of the modular field given by evaluation at the point in the upper half plane parameterizing the ground state φ.

Another important nuance is that the algebra A is no longer unital while $\mathcal{A}_\mathbb{Q}$ is a subalgebra of the algebra of unbounded multipliers of A. Just as an ordinary function need not be bounded to be integrable, so states can be evaluated on unbounded multipliers. In our case, the rational subalgebra $\mathcal{A}_\mathbb{Q}$ is not self-adjoint.

Finally the action of the symmetry group on the ground states is obtained via the action on states at positive temperature. Given a ground state, one warms it up below the critical temperature and acts on it by endomorphisms. When taking the limit to zero temperature of the resulting state, one obtains the corresponding transformed ground state. In our framework, the correct notion of ground states is given by a stronger form of the KMS_∞ condition, where we also require that these are weak limits of KMS_β states for $\beta \to \infty$.

We then consider the "dual" system of the GL_2-system, which describes the space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices (not up to scaling). The corresponding algebra is closely related to the modular Hecke algebra of [10]. As in the 1-dimensional case, where the corresponding space is compactified by removing the non-zero condition for the archimedean component of the adèle, the compactification of the two-dimensional system amounts to replacing the archimedean component $\mathrm{GL}_2(\mathbb{R})$ with matrices $M_2(\mathbb{R})$. This corresponds to the noncommutative compactification of modular curves considered in [37]. In terms of $\mathbb{Q}$-lattices this corresponds to degenerations to pseudo-lattices, as in [34].

It is desirable to have a concrete physical (experimental) system realizing the BC symmetry breaking phenomenon (as suggested in [43]). In fact, we shall show

that the explicit presentation of the BC algebra not only exhibits a strong analogy with phase states, as in the theory of optical coherence, but it also involves an action on them of a discrete scaling group, acting by integral multiplication of frequencies.

Acknowledgements. We are very grateful to Niranjan Ramachandran for many extremely useful conversations on class field theory and KMS states, that motivated the GL_2 system described here, whose relation to the theory of complex multiplication is being investigated in [13]. We thank Marcelo Laca for giving us an extensive update on the further developments on [5]. We benefited from visits of the first author to MPI and of the second author to IHES and we thank both institutions for their hospitality. The second author is partially supported by a Sofja Kovalevskaya Award of the Humboldt Foundation and the German Government.

1.2 Quantum Statistical Mechanics

In classical statistical mechanics a state is a probability measure μ on the phase space that assigns to each observable f an expectation value, in the form of an average

$$\int f\, d\mu. \tag{1.1}$$

In particular for a Hamiltonian system, the Gibbs canonical ensemble is a measure defined in terms of the Hamiltonian and the symplectic structure on the phase space. It depends on a parameter β, which is an inverse temperature, $\beta = 1/kT$ with k the Boltzmann constant. The Gibbs measure is given by

$$d\mu_G = \frac{1}{Z} e^{-\beta H} d\mu_{Liouville}, \tag{1.2}$$

normalized by $Z = \int e^{-\beta H} d\mu_{Liouville}$.

When passing to infinitely many degrees of freedom, where the interesting phenomena of phase transitions and symmetry breaking happen, the definition of the Gibbs states becomes more involved (*cf.* [45]). In the quantum mechanical framework, the analog of the Gibbs condition is given by the KMS condition at inverse temperature β ([17]). This is simpler in formulation than its classical counterpart, as it relies only on the involutive algebra A of observables and its time evolution $\sigma_t \in \mathrm{Aut}(A)$, and does not involve any additional structure like the symplectic structure or the approximation by regions of finite volume.

In fact, quantum mechanically, the observables form a C^*-algebra A, the Hamiltonian is the infinitesimal generator of the (pointwise norm continuous) one parameter group of automorphisms $\sigma_t \in \mathrm{Aut}(A)$, and the analog of a probability measure, assigning to every observable a certain average, is given by a *state*.

Definition 1.1 *A state on a C^*-algebra A is a linear form on A such that*

$$\varphi(1) = 1\,, \quad \varphi(a^* a) \geqq 0 \qquad \forall\, a \in A\,. \tag{1.3}$$

When the C^*-algebra A is non unital the condition $\varphi(1) = 1$ is replaced by $||\varphi|| = 1$ where

$$||\varphi|| := \sup_{x \in A, ||x|| \leq 1} |\varphi(x)|\,. \tag{1.4}$$

Such states are restrictions of states on the unital C^*-algebra $\tilde{A}$ obtained by adjoining a unit.

The evaluation $\varphi(a)$ gives the expectation value of the observable a in the statistical state φ. The Gibbs relation between a thermal state at inverse temperature $\beta = \frac{1}{kT}$ and the time evolution

$$\sigma_t \in \mathrm{Aut}(A) \tag{1.5}$$

is encoded by the KMS condition ([17]) which reads

$$\forall\, a, b \in A\,,\ \exists\, F \text{ bounded holomorphic in the strip } \{z \,|\, \mathrm{Im}\, z \in [0, \beta]\} \tag{1.6}$$

$$F(t) = \varphi(a\, \sigma_t(b)) \qquad F(t + i\beta) = \varphi(\sigma_t(b) a) \quad \forall t \in \mathbb{R}.$$

In the case of a system with finitely many quantum degrees of freedom, the algebra of observables is the algebra of operators in a Hilbert space $\mathcal{H}$ and the time evolution is given by $\sigma_t(a) = e^{itH}\, a\, e^{-itH}$, where H is a positive self-adjoint operator such that $\exp(-\beta H)$ is trace class for any $\beta > 0$. For such a system, the analog of (1.2) is

$$\varphi(a) = \frac{1}{Z}\, \mathrm{Tr}\left(a\, e^{-\beta H}\right) \qquad \forall a \in A, \tag{1.7}$$

with the normalization factor $Z = \mathrm{Tr}(\exp(-\beta H))$. It is easy to see that (1.7) satisfies the KMS condition (1.6) at inverse temperature β.

In the nonunital case, the KMS condition is defined in the same way by (1.6). Let $M(A)$ be the multiplier algebra of A and let $B \subset M(A)$ be the C^*-subalgebra of elements $x \in M(A)$ such that $t \mapsto \sigma_t(x)$ is norm continuous.

Proposition 1.2 *Any state φ on A admits a canonical extension to a state still noted φ on the multiplier algebra $M(A)$ of A. The canonical extension of a KMS state still satisfies the KMS condition on B.*

Proof. For the first statement we refer to [42]. The proof of the second statement illustrates a general density argument, where the continuity of $t \mapsto \sigma_t(x)$ is used to control the uniform continuity in the closed strip, in order to apply the Montel theorem of normal families. Indeed, by weak density of A in $M(A)$, one obtains a sequence of holomorphic functions, but one only controls their uniform continuity on smooth elements of B. $\square$

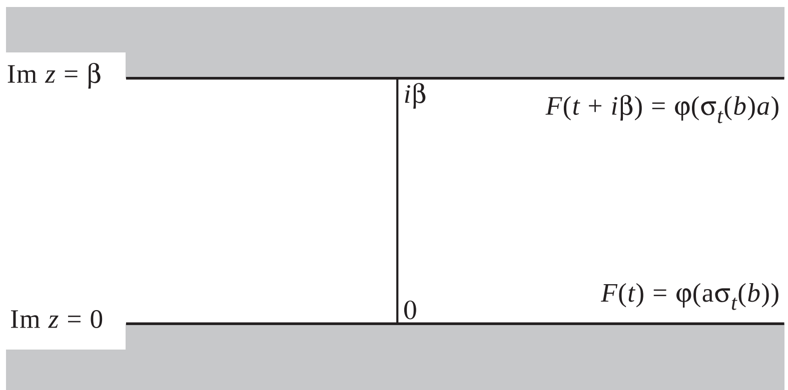

Figure 1.1: The KMS condition.

As we shall see, it will also be useful to extend whenever possible the integration provided by a state to unbounded multipliers of A.

In the unital case, for any given value of β, the set Σ_β of KMS_β states on A forms a convex compact Choquet simplex (possibly empty and in general infinite dimensional). In the nonunital case, given a σ_t-invariant subalgebra C of B, the set $\Sigma_\beta(C)$ of KMS_β states on C should be viewed as a compactification of the set of KMS_β states on A. The restriction from C to A maps $\Sigma_\beta(C)$ to KMS_β positive linear forms on A of norm less than or equal to one (quasi-states).

The typical pattern for a system with a single phase transition is that this simplex consists of a single point for $\beta \leqq \beta_c$ *i.e.* when the temperature is larger than the critical temperature T_c, and is non-trivial (of some higher dimension in general) when the temperature lowers. Systems can exhibit a more complex pattern of multiple phase transitions, where no KMS state exists above a certain temperature. The GL_2 system, which is the main object of study in this paper, will actually exhibit this more elaborate behavior.

We refer to the books ([6], [16]) for the general discussion of KMS states and phase transitions. The main technical point is that for finite β a β-KMS state is extremal iff the corresponding GNS representation is factorial. The decomposition into extremal β-KMS states is then the primary decomposition for a given β-KMS state.

At 0 temperature ($\beta = \infty$) the interesting notion is that of weak limit of β-KMS states for $\beta \to \infty$. It is true that such states satisfy a weak form of the KMS

condition. This can be formulated by saying that, for all $a, b \in A$, the function

$$F(t) = \varphi(a\, \sigma_t(b))$$

extends to a bounded holomorphic function in the upper half plane $\mathbb{H}$. This implies that, in the Hilbert space of the GNS representation of φ (*i.e.* the completion of A in the inner product $\varphi(a^* b)$), the generator H of the one-parameter group σ_t is a positive operator (positive energy condition). However, this condition is too weak in general to be interesting, as one sees by taking the trivial evolution ($\sigma_t = \mathrm{id}\,, \quad \forall t \in \mathbb{R}$). In this case any state fulfills it, while weak limits of β-KMS states are automatically tracial states. Thus, we shall define $\Sigma_{\beta=\infty}$ as the set of weak limit points of the sets Σ_β of β-KMS states for $\beta \to \infty$.

The framework for spontaneous symmetry breaking ([16]) involves a (compact) group of automorphisms $G \subset \mathrm{Aut}(A)$ of A commuting with the time evolution,

$$\sigma_t\, \alpha_g = \alpha_g\, \sigma_t \qquad \forall\, g \in G\,,\ t \in \mathbb{R}\,. \tag{1.8}$$

The group G is the symmetry group of the system, and the choice of an equilibrium state φ may break it to a smaller subgroup given by the isotropy group of φ

$$G_\varphi = \{g \in G\,,\ g\varphi = \varphi\}. \tag{1.9}$$

The group G acts on Σ_β for any β, hence on its extreme points $\mathcal{E}(\Sigma_\beta) = \mathcal{E}_\beta$. The unitary group $\mathcal{U}$ of the fixed point algebra of σ_t acts by inner automorphisms of the dynamical system (A, σ_t): for $u \in \mathcal{U}$,

$$(\mathrm{Ad}u)\,(x) := \; u\, x\, u^*\,, \quad \forall x \in A\,.$$

These *inner* automorphisms of (A, σ_t) act trivially on KMS_β states, as one checks using the KMS condition. This gives us the freedom to wipe out the group $\mathrm{Int}(A, \sigma_t)$ of inner symmetries and to define an action *modulo inner* of a group G on the system (A, σ_t) as a map

$$\alpha : G \to \mathrm{Aut}(A, \sigma_t)$$

fulfilling the condition

$$\alpha(g_1 g_2)\, \alpha(g_2)^{-1}\, \alpha(g_1)^{-1} \in \mathrm{Int}(A, \sigma_t)\,, \quad \forall g_j \in G\,.$$

Such an action gives an action of the group G on the set Σ_β of KMS_β states since the ambiguity coming from $\mathrm{Int}(A, \sigma_t)$ disappears in the action on Σ_β. In fact there is one more generalization of the above obvious notion of symmetries that we shall crucially need later – it involves actions by endomorphisms. This type of symmetry plays a key role in the theory of superselection sectors developed by Doplicher-Haag-Roberts (cf.[16], Chapter IV).

Definition 1.3 *An* endomorphism ρ *of the dynamical system* (A, σ_t) *is a* $*$-*homomorphism* $\rho : A \to A$ *commuting with* σ_t.

It follows then that $\rho(1) = e$ is an idempotent fixed by σ_t. Given a KMS$_\beta$ state φ the equality

$$\rho^*(\varphi) := Z^{-1}\,\varphi\circ\rho\,, \quad Z = \phi(e)$$

gives a KMS$_\beta$ state, provided that $\varphi(e) \neq 0$. Exactly as above for unitaries, consider an isometry

$$u \in A\,, \quad u^*\,u = 1$$

which is an eigenvector for σ_t, *i.e.* that fulfills, for some $\lambda \in \mathbb{R}_+^*$ $(\lambda \geq 1)$, the condition

$$\sigma_t(u) = \lambda^{it}\,u\,, \quad \forall t \in \mathbb{R}\,.$$

Then u defines an *inner* endomorphism Adu of the dynamical system (A, σ_t) by the equality

$$(\mathrm{Ad}u)\,(x) := \,u\,x\,u^*\,, \quad \forall x \in A\,,$$

and one obtains the following.

Proposition 1.4 *The inner endomorphisms of the dynamical system (A, σ_t) act trivially on the set of KMS$_\beta$ states,*

$$(\mathrm{Ad}u)^*(\varphi) = \varphi\,, \quad \forall \varphi \in \Sigma_\beta\,.$$

Proof. The KMS$_\beta$ condition shows that $\varphi(u\,u^*) = \lambda^{-\beta} > 0$ so that $(\mathrm{Ad}u)^*(\varphi)$ is well defined. The same KMS$_\beta$ condition applied to the pair $(x\,u^*,\,u)$ shows that $(\mathrm{Ad}u)^*(\varphi) = \varphi$. □

At 0 temperature ($\beta = \infty$) it is no longer true that the endomorphisms act directly on the set Σ_∞ of KMS$_\infty$ states, but one can use their action on KMS$_\beta$-states together with the "warming up" operation. This is defined as the map

$$W_\beta(\varphi)(x) = \,Z^{-1}\,\mathrm{Trace}(\pi(x)\,e^{-\beta\,H})\,, \quad \forall x \in A\,, \tag{1.10}$$

where H is the positive energy Hamiltonian, implementing the time evolution in the representation π associated to the KMS$_\infty$ state φ and $Z = \mathrm{Trace}(\,e^{-\beta\,H})$. Typically, W_β gives a bijection

$$W_\beta : \Sigma_\infty \to \Sigma_\beta,$$

for β larger than critical. Using the bijection W_β, one can transfer the action back to zero temperature states.

Another property of KMS states that we shall need later is the following functoriality. Namely, besides the obvious functoriality under pullback, discussed above, KMS states push forward under equivariant surjections, modulo normalization.

Proposition 1.5 *Let (A, σ_t) be a C^*-dynamical system (A separable) and J a norm closed two sided ideal of A globally invariant under σ_t. Let u_n be a quasi central approximate unit for J. For any KMS$_\beta$-state φ on (A, σ_t) the following sequence converges and defines a KMS$_\beta$ positive linear form on $(A/J, \sigma_t)$,*

$$\psi(x) = \lim_{n\to\infty} \varphi((1-u_n)\,x)\,, \quad \forall x \in A\,.$$

Proof. Let A'' be the double dual of A and $p \in A''$ the central open projection corresponding to the ideal J (cf. [42]). By construction the u_n converge weakly to p (cf. [42] 3.12.14) so the convergence follows as well as the positivity of ψ. By construction ψ vanishes on J. To get the KMS_β condition one applies (1.6) with $a = (1-u_n)\,x$, $b = y$ where y is a smooth element in A. Then one gets a bounded uniformly continuous sequence $F_n(z)$ of holomorphic functions in the strip $\{z \,|\, \mathrm{Im}\, z \in [0,\beta]\}$ with

$$F_n(t) = \varphi((1-u_n)\,x\,\sigma_t(y)) \qquad F_n(t+i\beta) = \varphi(\sigma_t(y)(1-u_n)\,x) \quad \forall t \in \mathbb{R}.$$

Using the Montel theorem on normal families and the quasi-central property of u_n one gets the KMS_β condition for ψ. $\square$

1.3 $\mathbb{Q}^{ab}$ and KMS states

We shall now describe an explicit system (*cf.* [4], [5]) that will make contact between the general framework above and arithmetic. The algebra $\mathcal{A}$ of this system is defined over the rationals,

$$\mathcal{A} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}\,, \tag{1.11}$$

where $\mathcal{A}_{\mathbb{Q}}$ is a $\mathbb{Q}$-algebra and is of countable (infinite) dimension as a vector space over $\mathbb{Q}$. The algebra $\mathcal{A}$ has a C^*-completion A and a natural time evolution σ_t.

To any vacuum state $\varphi \in \mathcal{E}_\infty$ for (A, σ_t) we attach the $\mathbb{Q}$-vector space of complex numbers,

$$V_\varphi := \{\varphi(a)\,;\ a \in \mathcal{A}_{\mathbb{Q}}\} \tag{1.12}$$

that is of countable dimension over $\mathbb{Q}$. It turns out that V_φ is included in algebraic numbers, so that one can act on these numbers by the Galois group

$$\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})\,. \tag{1.13}$$

The symmetry group G is the inverse limit with the profinite topology

$$G = \hat{\mathbb{Z}}^* = \varprojlim_n \mathrm{GL}_1(\mathbb{Z}/n\mathbb{Z}). \tag{1.14}$$

This can also be described as the quotient of the idèle class group of $\mathbb{Q}$ by the connected component of the identity,

$$G = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})/\mathbb{R}_+^* = C_{\mathbb{Q}}/D_{\mathbb{Q}}\,. \tag{1.15}$$

Here $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ denotes the adèles of $\mathbb{Q}$, namely $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$, with $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$.

The following amazing fact holds:

For any $\varphi \in \mathcal{E}_\infty$ and any $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the composition (1.16)

$\gamma \circ \varphi$ defined on $\mathcal{A}_{\mathbb{Q}}$ does extend to a *state* on $\mathcal{A}$.

What is "unreasonable" in this property defining "fabulous" states is that, though elements

$$\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \tag{1.17}$$

extend to automorphisms of $\mathbb{C}$, these are extremely discontinuous and not even Lebesgue measurable (except for $z \mapsto \bar{z}$), and certainly do not preserve positivity.

It follows from (1.16) that the composition $\varphi \mapsto \gamma \circ \varphi$ defines uniquely an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{E}_\infty$ and the equation

$$\gamma \circ \varphi = \varphi \circ g \tag{1.18}$$

gives a relation between Galois automorphisms and elements of G, *i.e.* idèle classes (1.15), which is in fact the class field theory isomorphism $C_{\mathbb{Q}}/D_{\mathbb{Q}} \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$.

Let us now concretely describe our system, consisting of the algebra $\mathcal{A}$ (defined over $\mathbb{Q}$) and of the time evolution σ_t.

The main conceptual steps involved in the construction of this algebra are:

- The construction, due to Hecke, of convolution algebras associated to double cosets on algebraic groups over the rational numbers;
- The existence of a canonical time evolution on a von Neumann algebra.

More concretely, while Hecke was considering the case of GL_2, where Hecke operators appear in the convolution algebra associated to the almost normal subgroup $\mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Q})$, the BC system arises from the Hecke algebra associated to the corresponding pair of parabolic subgroups.

Indeed, let P be the algebraic group "$ax+b$", *i.e.* the functor which to any abelian ring R assigns the group P_R of 2 by 2 matrices over R of the form

$$P_R = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \; ; \; a, b \in R\,, \; a \text{ invertible} \right\} . \tag{1.19}$$

By construction $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$ is an inclusion $\Gamma_0 \subset \Gamma$ of countable groups, where P_R^+ denotes the restriction to $a > 0$. This inclusion fulfills the following commensurability condition:

$$\text{The orbits of the left action of } \Gamma_0 \text{ on } \Gamma/\Gamma_0 \text{ are all } \textit{finite}. \tag{1.20}$$

For obvious reasons the same holds for orbits of Γ_0 acting on the right on $\Gamma_0\backslash\Gamma$.

The Hecke algebra $\mathcal{A}_{\mathbb{Q}} = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ is by definition the convolution algebra of functions of finite support

$$f : \Gamma_0\backslash\Gamma \to \mathbb{Q}\,, \tag{1.21}$$

which fulfill the Γ_0-invariance condition

$$f(\gamma\gamma_0) = f(\gamma) \qquad \forall\, \gamma \in \Gamma\,, \; \gamma_0 \in \Gamma_0 \tag{1.22}$$

so that f is really defined on $\Gamma_0\backslash\Gamma/\Gamma_0$. The convolution product is then given by

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0\backslash\Gamma} f_1(\gamma\gamma_1^{-1})f_2(\gamma_1)\,. \tag{1.23}$$

The time evolution appears from the analysis of the *regular representation* of the pair (Γ, Γ_0). It is trivial when Γ_0 is normal, or in the original case of Hecke, but it becomes interesting in the parabolic case, due to the lack of unimodularity of the parabolic group, as will become clear in the following.

The regular representation

$$(\pi(f)\xi)(\gamma) = \sum_{\Gamma_0\backslash\Gamma} f(\gamma\gamma_1^{-1})\xi(\gamma_1) \tag{1.24}$$

in the Hilbert space

$$\mathcal{H} = \ell^2(\Gamma_0\backslash\Gamma) \tag{1.25}$$

extends to the complexification

$$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \tag{1.26}$$

of the above algebra, which inherits from this representation the involution $a \mapsto a^*$, uniquely defined so that $\pi(a^*) = \pi(a)^*$ (the Hilbert space adjoint), namely

$$f^*(\gamma) := \overline{f(\gamma^{-1})} \qquad \forall\, \gamma \in \Gamma_0\backslash\Gamma/\Gamma_0. \tag{1.27}$$

It happens that the time evolution (*cf.* [54]) of the von Neumann algebra generated by $\mathcal{A}$ in the regular representation restricts to the dense subalgebra $\mathcal{A}$. This implies that there is a uniquely determined time evolution $\sigma_t \in \mathrm{Aut}(\mathcal{A})$, such that the state φ_1 given by

$$\varphi_1(f) = \langle \pi(f)\varepsilon_e, \varepsilon_e \rangle \tag{1.28}$$

is a KMS_1 state *i.e.* a KMS state at inverse temperature $\beta = 1$. Here ε_e is the cyclic and separating vector for the regular representation given by the left coset $\{\Gamma_0\} \in \Gamma_0\backslash\Gamma$.

Explicitly, one gets the following formula for the time evolution:

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma) \qquad \forall \gamma \in \Gamma_0\backslash\Gamma/\Gamma_0\,, \tag{1.29}$$

where the integer valued functions L and R on the double coset space are given respectively by

$$L(\gamma) = \text{ Cardinality of left } \Gamma_0 \text{ orbit of } \gamma \text{ in } \Gamma/\Gamma_0\,, \quad R(\gamma) = L(\gamma^{-1})\,. \tag{1.30}$$

Besides the conceptual description given above, the algebra $\mathcal{A}_{\mathbb{Q}}$ also has a useful explicit presentation in terms of generators and relations (*cf.* [5] §4, Prop.18). We recall it here, in the slightly simplified version of [21], Prop.24.

Proposition 1.6 *The algebra $\mathcal{A}_{\mathbb{Q}}$ is generated by elements μ_n, $n \in \mathbb{N}^\times$ and $e(r)$, for $r \in \mathbb{Q}/\mathbb{Z}$, satisfying the relations*

- $\mu_n^* \mu_n = 1$, *for all* $n \in \mathbb{N}^\times$,
- $\mu_k \mu_n = \mu_{kn}$, *for all* $k, n \in \mathbb{N}^\times$,
- $e(0) = 1$, $e(r)^* = e(-r)$, *and* $e(r)e(s) = e(r+s)$, *for all* $r, s \in \mathbb{Q}/\mathbb{Z}$,
- *For all* $n \in \mathbb{N}^\times$ *and all* $r \in \mathbb{Q}/\mathbb{Z}$,

$$\mu_n \, e(r) \, \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s). \tag{1.31}$$

In this form the time evolution preserves pointwise the subalgebra $R_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ generated by the $e(r)$ and acts on the μ_n's as

$$\sigma_t(\mu_n) = n^{it} \, \mu_n.$$

The Hecke algebra considered above admits an automorphism α, $\alpha^2 = 1$ whose fixed point algebra is the Hecke algebra of the pair $P_{\mathbb{Z}} \subset P_{\mathbb{Q}}$. The latter admits an equivalent description[2], from the pair

$$(P_R, P_{\mathbb{A}_f}),$$

where R is the maximal compact subring of the ring of finite adèles

$$\mathbb{A}_f = \prod_{\text{res}} \mathbb{Q}_p \, . \tag{1.32}$$

This adèlic description displays, as a natural symmetry group, the quotient G of the idèle class group of $\mathbb{Q}$ by the connected component of identity (1.15).

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $\mathbb{Q}^{ab} \subset \overline{\mathbb{Q}}$ be the maximal abelian extension of $\mathbb{Q}$. Let $r \mapsto \zeta_r$ be a (non-canonical) isomorphism of $\mathbb{Q}/\mathbb{Z}$ with the multiplicative group of roots of unity inside $\mathbb{Q}^{ab}$.

We can now state the basic result that gives content to the relation between phase transition and arithmetic (BC [5]):

Theorem 1.7 *1. For $0 < \beta \leqq 1$ there exists a unique* KMS_β *state φ_β for the above system. Its restriction to $R_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \subset \mathcal{A}$ is given by*

$$\varphi_\beta \left(e(a/b) \right) = b^{-\beta} \prod_{p \,\text{prime},\, p|b} \left(\frac{1 - p^{\beta-1}}{1 - p^{-1}} \right) . \tag{1.33}$$

[2]This procedure holds more generally (*cf.* [47] [48]) for arbitrary almost normal inclusions (Γ_0, Γ).

2. *For $\beta > 1$ the extreme* KMS_β *states are parameterized by embeddings* $\rho : \mathbb{Q}^{ab} \to \mathbb{C}$ *and*

$$\varphi_{\beta,\rho}\left(e(a/b)\right) = Z(\beta)^{-1} \sum_{n=1}^{\infty} n^{-\beta} \rho\left(\zeta_{a/b}^{n}\right), \tag{1.34}$$

where the partition function $Z(\beta) = \zeta(\beta)$ is the Riemann zeta function.

3. *For $\beta = \infty$, the Galois group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts by composition on $\mathcal{E}_\infty$. The action factors through the abelianization* $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$*, and the class field theory isomorphism $\theta : G \to \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ intertwines the actions,*

$$\alpha \circ \varphi = \varphi \circ \theta^{-1}(\alpha), \qquad \alpha \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}).$$

1.4 Further Developments

The main theorem of class field theory provides a classification of finite abelian extensions of a local or global field K in terms of subgroups of a locally compact abelian group canonically associated to the field. This is the multiplicative group $K^* = \mathrm{GL}_1(K)$ in the local nonarchimedean case, while in the global case it is the quotient C_K/D_K of the idèle class group C_K by the connected component of the identity. The construction of the group C_K is at the origin of the theory of idèles and adèles.

Hilbert's 12th problem can be formulated as the question of providing an explicit set of generators of the maximal abelian extension K^{ab} of a number field K, inside an algebraic closure $\bar{K}$, and of the action of the Galois group $\mathrm{Gal}(K^{ab}/K)$. The typical example where this is achieved, which motivated Hilbert's formulation of the explicit class field theory problem, is the Kronecker–Weber case: the construction of the maximal abelian extension of $\mathbb{Q}$. In this case the torsion points of $\mathbb{C}^*$ (roots of unity) generate $\mathbb{Q}^{ab} \subset \mathbb{C}$.

Remarkably, the only other case for number fields where this program has been carried out completely is that of imaginary quadratic fields, where the construction relies on the theory of elliptic curves with complex multiplication (*cf. e.g.* [53]). Generalizations to other number fields involve other remarkable problems in number theory like the Stark conjectures. Recent work of Manin [34] [35] suggests a close relation between the real quadratic case and noncommutative geometry.

To better appreciate the technical difficulties underlying any attempt to address the Hilbert 12th problem of explicit class field theory via the BC approach, in view of the problem of fabulous states that we shall formulate in §1.5, we first summarize briefly the state of the art (to this moment and to our knowledge) in the study of C^*-dynamical systems with phase transitions associated to number fields.

Some progress from the original BC paper followed in various directions, and some extensions of the BC construction to other global fields (number fields and function fields) were obtained. Harari and Leichtnam [19] produced a C^*-dynamical system with phase transition for function fields and algebraic number fields. In the number fields case a localization is used in order to deal with lack of unique factorization into primes, and this changes finitely many Euler factors in the zeta function. The construction is based on the inclusion $\mathcal{O} \rtimes 1 \subset K \rtimes K^*$, where $\mathcal{O}$ is the ring of integers of K. The symmetry group G of (1.15) is replaced by the group

$$G = \hat{\mathcal{O}}^* = \mathrm{GL}_1(\hat{\mathcal{O}}),$$

with $\hat{\mathcal{O}}$ the profinite completion of the ring of integers $\mathcal{O}$. Their result on fixed point algebras of compact group actions shows that the contribution of the group of units $\mathcal{O}^*$ can be factored out in the cases when this group is finite. There is a group homomorphism $s : G \to C_K/D_K$, but it is in general neither injective nor surjective, hence, even in the case of imaginary quadratic fields, the construction does not capture the action of the Galois group $\mathrm{Gal}(K^{ab}/K)$, except in the very special class number one case.

P. Cohen gave in [8] a construction of a C^*-dynamical system associated to a number field K, which has spontaneous symmetry breaking and recovers the full Dedekind zeta function as partition function. The main point of her approach is to involve the semigroup of all ideals rather than just the principal ideals used in other approaches as the replacement of the semi-group of positive integers involved in BC. Still, the group of symmetries is $G = \hat{\mathcal{O}}^*$ and not the desired C_K/D_K.

Typically, the extensions of the number field K obtained via these constructions are given by roots of unity, hence they do not recover the maximal abelian extension.

The Hecke algebra of the inclusion $\mathcal{O} \rtimes 1 \subset K \rtimes K^*$ for an arbitrary algebraic number field K was also considered by Arledge, Laca, and Raeburn in [1], where they discuss its structure and representations, but not the problem of KMS states.

Further results on this Hecke algebra have been announced by Laca and van Frankenhuysen [27]: they obtain some general results on the structure and representations for all number fields, while they analyze the structure of KMS states only for the class number one case. In this case, their announced result is that there are enough ground states to support a transitive free action of $\mathrm{Gal}(K^{ab}/K)$ (up to a copy of $\{\pm 1\}$ for each real embedding). However, it appears that the construction does not give embeddings of K^{ab} as actual values of the ground states on the Hecke algebra over K, hence it does not seem suitable to treat the class field theory problem of providing explicit generators of K^{ab}.

The structure of the Hecke algebra of the inclusion $\mathcal{O} \rtimes \mathcal{O}^* \subset K \rtimes K^*$ was further clarified by Laca and Larsen in [23], using a decomposition of the Hecke algebra of a semidirect product as the cross product of the Hecke algebra of an intermediate (smaller) inclusion by an action of a semigroup.

The original BC algebra was also studied in much greater details in several following papers. It was proved by Brenken in [7] and by Laca and Raeburn in [25] that the BC algebra can be written as a semigroup cross product. Brenken also discusses the case of Hecke algebras from number fields of the type considered in [25, 1].

Laca then re-derived the original BC result from the point of view of semigroup cross products in [21]. This allows for significant simplifications of the argument in the case of $\beta > 1$, by looking at the conditional expectations and the KMS condition at the level of the "predual" (semigroup) dynamical system. A further simplification of the original phase transition theorem of BC was given by Neshveyev in [40], via a direct argument for ergodicity, which implies uniqueness of the KMS states for $0 \le \beta \le 1$.

The BC algebra can also be realized as a full corner in the cross product of the finite adèles by the multiplicative rationals, as was shown by Laca in [22], by dilating the semigroup action to a minimal full group action. Laca and Raeburn used the dilation results of [22] to calculate explicitly the primitive (and maximal) ideal spaces of the BC algebra as well as of the cross product of the full adèles by the action of the multiplicative rationals.

Using the cross product description of the BC algebra, Leichtnam and Nistor computed Hochschild, cyclic, and periodic cyclic homology groups of the BC algebra, by computing the corresponding groups for the C^*-dynamical system algebras arising from the action of $\mathbb{Q}^*$ on the adèles of $\mathbb{Q}$. The calculation for the BC algebra then follows by taking an increasing sequence of smooth subalgebras and an inductive limit over certain Morita equivalent subalgebras.

Further results related to aspects of the BC construction and generalizations can be found in [3], [15], [24], [29], [30], [55].

1.5 Fabulous States

Given a number field K, we let $\mathbb{A}_K$ denote the adèles of K and $J_K = \mathrm{GL}_1(\mathbb{A}_K)$ be the group of idèles of K. We write C_K for the group of idèles classes $C_K = J_K/K^*$ and D_K for the connected component of the identity in C_K.

If we remain close to the spirit of the Hilbert 12th problem, we can formulate a general question, aimed at extending the results of [5] to other number fields K. Given a number field K, with a choice of an embedding $K \subset \mathbb{C}$, the "problem of fabulous states" consists in constructing a C^*-dynamical system (A, σ_t) and an "arithmetic" subalgebra $\mathcal{A}$, which satisfy the following properties:

1. The idèles class group $G = C_K/D_K$ acts by symmetries on (A, σ_t) preserving the subalgebra $\mathcal{A}$.
2. The states $\varphi \in \mathcal{E}_\infty$, evaluated on elements of $\mathcal{A}$, satisfy:
   - $\varphi(a) \in \bar{K}$, the algebraic closure of K in $\mathbb{C}$;

- the elements of $\{\varphi(a) : a \in \mathcal{A}\}$, for $\varphi \in \mathcal{E}_\infty$ generate K^{ab}.

3. The class field theory isomorphism

$$\theta : C_K/D_K \xrightarrow{\simeq} \mathrm{Gal}(K^{ab}/K)$$

intertwines the actions,

$$\alpha \circ \varphi = \varphi \circ \theta^{-1}(\alpha), \tag{1.35}$$

for all $\alpha \in \mathrm{Gal}(K^{ab}/K)$ and for all $\varphi \in \mathcal{E}_\infty$.

Notice that, with this formulation, the problem of the construction of fabulous states is intimately related to Hilbert's 12th problem. This question will be pursued in [13].

We shall construct here a system which is the analog of the BC system for $\mathrm{GL}_2(\mathbb{Q})$ instead of $\mathrm{GL}_1(\mathbb{Q})$. This will extend the results of [5] to this non-abelian GL_2 case and will exhibit many new features which have no counterpart in the abelian case. Our construction involves the explicit description of the automorphism group of the modular field, [51]. The construction of fabulous states for imaginary quadratic fields, which will be investigated with N. Ramachandran in [13], involves specializing the GL_2 system to a subsystem compatible with complex multiplication in a given imaginary quadratic field.

The construction of the GL_2 system gives a C^*-dynamical system (A, σ_t) and an involutive subalgebra $\mathcal{A}_{\mathbb{Q}}$ defined over $\mathbb{Q}$, satisfying the following properties:

- The quotient group $S := \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$ of the finite adèlic group of GL_2 acts as symmetries of the dynamical system (A, σ_t) preserving the subalgebra $\mathcal{A}_{\mathbb{Q}}$.
- For generic $\varphi \in \mathcal{E}_\infty$, the values $\{\varphi(a) \in \mathbb{C} : a \in \mathcal{A}_{\mathbb{Q}}\}$ generate a subfield $F_\varphi \subset \mathbb{C}$ which is an extension of $\mathbb{Q}$ of transcendence degree 1.
- For generic $\varphi \in \mathcal{E}_\infty$, there exists an isomorphism

$$\theta : S \xrightarrow{\simeq} \mathrm{Gal}(F_\varphi/\mathbb{Q})$$

which intertwines the actions

$$\alpha \circ \varphi = \varphi \circ \theta^{-1}(\alpha), \tag{1.36}$$

for all $\alpha \in \mathrm{Gal}(F_\varphi/\mathbb{Q})$.

There are a number of important nuances between the abelian case above and the non-abelian one. For instance, in the abelian case the field generated by $\varphi(\mathcal{A})$ does not depend on the choice of $\varphi \in \mathcal{E}_\infty$ and the isomorphism θ is also independent of φ. This no longer holds in the non-abelian case, as is clear from the presence of inner automorphisms of the symmetry group S. Also, in

the latter case, the action of S on $\mathcal{E}_\infty$ is not transitive and the corresponding invariant of the orbit of φ under S is the subfield $F_\varphi \subset \mathbb{C}$. Another important nuance is that the algebra A is no longer unital while $\mathcal{A}_\mathbb{Q}$ is an algebra of unbounded multipliers of A. Finally, the symmetries require the full framework of endomorphisms as explained above in §1.2.

1.6 The subalgebra $\mathcal{A}_\mathbb{Q}$ and Eisenstein Series

In this section we shall recast the BC algebra in terms of the trigonometric analog of the Eisenstein series, following the analogy developed by Eisenstein and Kronecker between trigonometric and elliptic functions, as outlined by A.Weil in [56].

This will be done by first giving a geometric interpretation in terms of $\mathbb{Q}$-lattices of the noncommutative space X whose algebra of continuous functions $C(X)$ is the BC C^*-algebra. The space X is by construction the quotient of the Pontrjagin dual of the abelian group $\mathbb{Q}/\mathbb{Z}$ by the equivalence relation generated by the action by multiplication of the semi-group $\mathbb{N}^\times$.

Let

$$R = \prod_p \mathbb{Z}_p$$

be the compact ring product of the rings $\mathbb{Z}_p$ of p-adic integers. It is the maximal compact subring of the locally compact ring of finite adèles

$$\mathbb{A}_f = \prod_{\text{res}} \mathbb{Q}_p$$

We recall the following standard fact

Proposition 1.8
- *The inclusion $\mathbb{Q} \subset \mathbb{A}_f$ gives an isomorphism of abelian groups*

$$\mathbb{Q}/\mathbb{Z} = \mathbb{A}_f/R\,.$$

- *The following map is an isomorphism of compact rings*

$$j:\ R \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})\,, \quad j(a)(x) = a\,x\,, \quad \forall x \in \mathbb{A}_f/R\,, \quad \forall a \in R\,.$$

We shall use j from now on to identify R with $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Note that by construction $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is endowed with the topology of pointwise convergence. It is identified with $\varprojlim \mathbb{Z}/N\mathbb{Z}$ using the restriction to N-torsion elements.

For every $r \in \mathbb{Q}/\mathbb{Z}$ one gets a function $e(r) \in C(R)$ by,

$$e(r)(\rho) := \exp 2\pi i \rho(r) \quad \forall \rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

and this gives the identification of R with the Pontrjagin dual of $\mathbb{Q}/\mathbb{Z}$ and of $C(R)$ with the group C^*-algebra $C^*(\mathbb{Q}/\mathbb{Z})$.

One can then describe the BC C^*-algebra as the cross product of $C(R)$ by the semigroup action of $\mathbb{N}^\times$ as follows. For each integer $n \in \mathbb{N}^\times$ we let $nR \subset R$ be the range of multiplication by n. It is an open and closed subset of R whose characteristic function π_n is a projection $\pi_n \in C(R)$. One has by construction

$$\pi_n \, \pi_m = \pi_{n \vee m} \,, \quad \forall n \,, \; m \in \mathbb{N}^\times$$

where $n \vee m$ denotes the lowest common multiple of n and m.
The semigroup action of $\mathbb{N}^\times$ on $C(R)$ corresponds to the isomorphism

$$\alpha_n(f)(\rho) := f(n^{-1} \, \rho) \,, \quad \forall \rho \in nR \,. \tag{1.37}$$

of $C(R)$ with the reduced algebra $C(R)_{\pi_n}$ of $C(R)$ by the projection π_n. In the BC algebra one has

$$\mu_n \, f \, \mu_n^* = \alpha_n(f) \,, \quad \forall f \in C(R). \tag{1.38}$$

There is an equivalent description of the BC algebra in terms of the étale groupoid G of pairs (r, ρ), where $r \in \mathbb{Q}_+^*$, $\rho \in R$ and $r \, \rho \in R$. The composition in G is given by

$$(r_1, \rho_1) \circ (r_2, \rho_2) = (r_1 \, r_2, \rho_2) \,, \quad \text{if } r_2 \, \rho_2 = \rho_1 \,, \tag{1.39}$$

and the convolution of functions by

$$f_1 * f_2(r, \rho) := \sum f_1(rs^{-1}, s \, \rho) \, f_2(s, \rho) \,, \tag{1.40}$$

while the adjoint of f is

$$f^*(r, \rho) := \overline{f(r^{-1}, r \, \rho)} \,. \tag{1.41}$$

All of this is implicit in ([5]) and has been amply described in the subsequent papers mentioned in §1.4. In the description above, μ_n is given by the function $\mu_n(r, \rho)$ which vanishes unless $r = n$ and is equal to 1 for $r = n$. The time evolution is given by

$$\sigma_t(f)(r, \rho) := r^{it} \, f(r, \rho) \,, \quad \forall f \in C^*(G) \,. \tag{1.42}$$

We shall now describe a geometric interpretation of this groupoid G in terms of commensurability of $\mathbb{Q}$-lattices. In particular, it will pave the way to the generalization of the BC system to higher dimensions. The basic simple geometric objects are $\mathbb{Q}$-lattices in $\mathbb{R}^n$, defined as follows.

Definition 1.9 *A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}^n$, and $\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$ an homomorphism of abelian groups.*

Two lattices Λ_j in $\mathbb{R}^n$ are commensurable iff their intersection $\Lambda_1 \cap \Lambda_2$ is of finite index in Λ_j. Their sum $\Lambda = \Lambda_1 + \Lambda_2$ is then a lattice and, given two homomorphisms of abelian groups $\phi_j : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q} \, \Lambda_j/\Lambda_j$, the difference $\phi_1 - \phi_2$ is well defined modulo $\Lambda = \Lambda_1 + \Lambda_2$.
Notice that in Definition 1.9 the homomorphism ϕ, in general, is not an isomorphism.

Definition 1.10 *A $\mathbb{Q}$-lattice (Λ, ϕ) is* invertible *if the map ϕ is an isomorphism of abelian groups.*

We consider a natural equivalence relation on the set of $\mathbb{Q}$-lattices defined as follows.

Proposition 1.11 *The following defines an equivalence relation called* commensurability *between $\mathbb{Q}$-lattices: (Λ_1, ϕ_1), (Λ_2, ϕ_2) are commensurable iff Λ_j are commensurable and $\phi_1 - \phi_2 = 0$ modulo $\Lambda = \Lambda_1 + \Lambda_2$.*

Proof. Indeed, let (Λ_j, ϕ_j) be three $\mathbb{Q}$-lattices and assume commensurability between the pairs $(1, 2)$ and $(2, 3)$. Then the lattices Λ_j are commensurable and are of finite index in $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3$. One has $\phi_1 - \phi_2 = 0$ modulo Λ, $\phi_2 - \phi_3 = 0$ modulo Λ and thus $\phi_1 - \phi_3 = 0$ modulo Λ. But $\Lambda' = \Lambda_1 + \Lambda_3$ is of finite index in Λ and thus $\phi_1 - \phi_3$ gives a group homomorphism

$$\mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \Lambda/\Lambda'$$

which is zero since $\mathbb{Q}^n/\mathbb{Z}^n$ is infinitely divisible and Λ/Λ' is finite. This shows that $\phi_1 - \phi_3 = 0$ modulo $\Lambda' = \Lambda_1 + \Lambda_3$ and hence that the pair $(1, 3)$ is commensurable. $\square$

Notice that every $\mathbb{Q}$-lattice in $\mathbb{R}$ is uniquely of the form

$$(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho)\,, \quad \lambda > 0, \tag{1.43}$$

with $\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z},\, \mathbb{Q}/\mathbb{Z}) = R$.

Proposition 1.12 *The map*

$$\gamma(r, \rho) = ((r^{-1}\,\mathbb{Z},\, \rho)\,, (\mathbb{Z}, \rho))\,, \quad \forall (r, \rho) \in G\,,$$

defines an isomorphism of locally compact étale groupoids between G and the quotient $\mathcal{R}\,/\,\mathbb{R}_+^*$ of the equivalence relation $\mathcal{R}$ of commensurability on the space of $\mathbb{Q}$-lattices in $\mathbb{R}$ by the natural scaling action of $\mathbb{R}_+^*$.

Proof. First since $r\,\rho \in R$ the pair $(r^{-1}\,\mathbb{Z},\, \rho) = r^{-1}(\mathbb{Z},\, r\,\rho)$ is a $\mathbb{Q}$-lattice and is commensurable to $(\mathbb{Z}, \rho)$. Thus, the map γ is well defined. Using the identification (1.43), we see that the restriction of γ to the objects $G^{(0)}$ of G is an isomorphism of R with the quotient of the space of $\mathbb{Q}$-lattices in $\mathbb{R}$ by the natural scaling action of $\mathbb{R}_+^*$. The freeness of this action shows that the quotient $\mathcal{R}\,/\,\mathbb{R}_+^*$ is still a groupoid, and one has

$$\gamma(r_1, \rho_1) \circ \gamma(r_2, \rho_2) = \gamma(r_1\,r_2, \rho_2) \quad \text{if } r_2\,\rho_2 = \rho_1\,.$$

Finally, up to scaling, every element of $\mathcal{R}$ is of the form

$$((r^{-1}\,\mathbb{Z},\, r^{-1}\,\rho')\,, (\mathbb{Z}, \rho))$$

where both ρ' and ρ are in R and $r = \frac{a}{b} \in \mathbb{Q}_+^*$. Moreover since $r^{-1}\rho' = \rho$ modulo $\frac{1}{a}\mathbb{Z}$ one gets $a\rho - b\rho' = 0$ and $r^{-1}\rho' = \rho$. Thus γ is surjective and is an isomorphism. $\Box$

This geometric description of the BC algebra allows us to generate in a natural manner a rational subalgebra which will generalize to the two dimensional case. In particular the algebra $C(R)$ can be viewed as the algebra of homogeneous functions of "weight 0" on the space of $\mathbb{Q}$-lattices for the natural scaling action of the multiplicative group $\mathbb{R}_+^*$ where weight k means

$$f(\lambda\Lambda, \lambda\phi) = \lambda^{-k} f(\Lambda, \phi), \quad \forall \lambda \in \mathbb{R}_+^* .$$

We let the function $c(\Lambda)$ be the multiple of the covolume $|\Lambda|$ of the lattice, specified by

$$2\pi i\, c(\mathbb{Z}) = 1 \tag{1.44}$$

The function c is homogeneous of weight -1 on the space of $\mathbb{Q}$-lattices. For $a \in \mathbb{Q}/\mathbb{Z}$, we then define a function $e_{1,a}$ of weight 0 by

$$e_{1,a}(\Lambda, \phi) = c(\Lambda) \sum_{y \in \Lambda + \phi(a)} y^{-1}, \tag{1.45}$$

where one uses Eisenstein summation *i.e.* $\lim_{N\to\infty} \sum_{-N}^{N}$ when $\phi(a) \neq 0$ and one lets $e_{1,a}(\Lambda, \phi) = 0$ when $\phi(a) = 0$.

The main result of this section is the following

Theorem 1.13 • *The $e_{1,a}$, $a \in \mathbb{Q}/\mathbb{Z}$ generate $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$.*

- *The rational algebra $\mathcal{A}_{\mathbb{Q}}$ is the subalgebra of $A = C^*(G)$ generated by the $e_{1,a}$, $a \in \mathbb{Q}/\mathbb{Z}$ and the μ_n, μ_n^*.*

We define more generally for each weight $k \in \mathbb{N}$ and each $a \in \mathbb{Q}/\mathbb{Z}$ a function $\epsilon_{k,a}$ on the space of $\mathbb{Q}$-lattices in $\mathbb{R}$ by

$$\epsilon_{k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-k}. \tag{1.46}$$

This is well defined provided $\phi(a) \neq 0$. For $\phi(a) = 0$ we let

$$\epsilon_{k,a}(\Lambda, \phi) = \lambda_k\, c(\Lambda)^{-k}, \tag{1.47}$$

where we shall fix the constants λ_k below in (1.50). The function $\epsilon_{k,a}$ has weight k *i.e.* it satisfies the homogeneity condition

$$\epsilon_{k,a}(\lambda\Lambda, \lambda\phi) = \lambda^{-k} \epsilon_{k,a}(\Lambda, \phi), \quad \forall \lambda \in \mathbb{R}_+^* .$$

When $a = \frac{b}{N}$ the function $\epsilon_{k,a}$ has level N in that it only uses the restriction ϕ_N of ϕ to N-torsion points of $\mathbb{Q}/\mathbb{Z}$,

$$\phi_N : \frac{1}{N}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{N}\Lambda/\Lambda.$$

The products

$$e_{k,a} := c^k \, \epsilon_{k,a} \tag{1.48}$$

are of weight 0 and satisfy two types of relations.

The first relations are multiplicative and express $e_{k,a}$ as a polynomial in $e_{1,a}$,

$$e_{k,a} = P_k(e_{1,a}) \tag{1.49}$$

where the P_k are the polynomials with rational coefficients uniquely determined by the equalities

$$P_1(u) = u\,, \quad P_{k+1}(u) = \frac{1}{k}(u^2 - \frac{1}{4})\,\partial_u P_k(u)\,.$$

This follows for $\phi(a) \notin \Lambda$ from the elementary formulas for the trigonometric analog of the Eisenstein series ([56] Chapter II). Since $e_{1,a}(\Lambda, \phi) = 0$ is the natural choice for $\phi(a) \in \Lambda$, the validity of (1.49) uniquely dictates the choice of the normalization constants λ_k of (1.47). One gets

$$\lambda_k = P_k(0) = (2^k - 1)\,\gamma_k\,, \tag{1.50}$$

where $\gamma_k = 0$ for odd k and $\gamma_{2j} = (-1)^j \, \dfrac{B_j}{(2j)!}$ with $B_j \in \mathbb{Q}$ the Bernoulli numbers. Equivalently,

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} - \sum_1^\infty \gamma_{2j}\, x^{2j}.$$

One can express the $e_{k,a}$ as $\mathbb{Q}$-linear combinations of the generators $e(r)$. We view $e(r)$ as the function on $\mathbb{Q}$-lattices which assigns to $(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho)$, $\lambda > 0$, the value

$$e(r)(\Lambda, \phi) := \exp 2\pi i \rho(r).$$

One then has

Lemma 1.14 *Let $a \in \mathbb{Q}/\mathbb{Z}$, and $n > 0$ with $n\,a = 0$. Then*

$$e_{1,a} = \sum_{k=1}^{n-1} (\frac{k}{n} - \frac{1}{2})\, e(k\,a). \tag{1.51}$$

Proof. We evaluate both sides on $(\Lambda, \phi) = (\lambda\,\mathbb{Z}, \lambda\,\rho)$, $\lambda > 0$. Both sides only depend on the restriction $x \mapsto c\,x$ of ρ to n-torsion elements of $\mathbb{Q}/\mathbb{Z}$ which we write as multiplication by $c \in \mathbb{Z}/n\mathbb{Z}$. Let $a = \frac{b}{n}$. If $bc = 0(n)$ then $\phi(a) = 0$ and

both sides vanish since $e(k\,a)(\Lambda,\phi) = \exp 2\pi i(\frac{kbc}{n}) = 1$ for all k. If $bc \neq 0(n)$ then $\phi(a) \neq 0$ and the left side is $\frac{1}{2}(U+1)/(U-1)$ where $U = \exp 2\pi i\frac{bc}{n}$, $U^n = 1$, $U \neq 1$. The right hand side is

$$\sum_{k=1}^{n-1} (\frac{k}{n} - \frac{1}{2})\, U^k \,,$$

which gives $\frac{1}{2}(U+1)$ after multiplication by $U-1$. $\Box$

This last equality shows that $e_{1,a}$ is (one half of) the Cayley transform of $e(a)$ with care taken where $e(a)-1$ fails to be invertible. In particular while $e(a)$ is unitary, $e_{1,a}$ is skew-adjoint,

$$e_{1,a}^* = \,-\,e_{1,a}\,.$$

We say that a $\mathbb{Q}$-lattice (Λ,ϕ) is divisible by an integer $n \in \mathbb{N}$ when $\phi_n = 0$. We let π_n be the characteristic function of the set of $\mathbb{Q}$-lattices divisible by n. It corresponds to the characteristic function of $n\,R \subset R$.
Let $N > 0$ and (Λ,ϕ) a $\mathbb{Q}$-lattice with $\phi_N(a) = c\,a$ for $c \in \mathbb{Z}/N\mathbb{Z}$. The order of the kernel of ϕ_N is $m = \gcd(N,c)$. Also a divisor $b|N$ divides (Λ,ϕ) iff it divides c. Thus for any function f on $\mathbb{N}^*$ one has

$$\sum_{b|N} f(b)\,\pi_b(\Lambda,\phi) = \sum_{b|\gcd(N,c)} f(b)\,,$$

which allows one to express any function of the order $m = \gcd(N,c)$ of the kernel of ϕ_N in terms of the projections π_b , $b|N$. In order to obtain the function $m \mapsto m^j$ we let

$$f_j(n) := \sum_{d|n} \mu(d)(n/d)^j\,,$$

where μ is the Möbius function so that

$$f_j(n) = \,n^j \prod_{p\,\mathrm{prime},\,p|n} (1-p^{-j})\,.$$

Notice that f_1 is the Euler totient function and that the ratio $f_{-\beta+1}/f_1$ gives the r.h.s. of (1.33) in Theorem 1.7.
The Möbius inversion formula gives

$$\sum_{b|N} f_j(b)\,\pi_b(\Lambda,\phi) = m^j\,, \quad m = \gcd(N,c)\,. \tag{1.52}$$

We can now write division relations fulfilled by the functions (1.48).

Lemma 1.15 *Let $N > 0$ then*

$$\sum_{N\,a=0} e_{k,a} = \gamma_k \sum_{d|N} ((2^k-2)\,f_1(d) + N^k f_{-k+1}(d))\,\pi_d. \tag{1.53}$$

Proof. For a given $\mathbb{Q}$-lattice (Λ,ϕ) with $\operatorname{Ker}\phi_N$ of order $m|N$, $N = m\,d$, the result follows from

$$\sum_{N\,a=0} \epsilon_{k,a}(\Lambda,\phi) = m \sum_{y\in\frac{1}{d}\Lambda\setminus\Lambda} y^{-k} + m\,(2^k-1)\,\gamma_k\,c^{-k}(\Lambda) = m\,(d^{\,k}+2^k-2)\gamma_k\,c^{-k}$$

together with (1.52) applied for $j=1$ and $j=1-k$. $\Box$

The semigroup action of $\mathbb{N}^\times$ is given on functions of $\mathbb{Q}$-lattices by the endomorphisms

$$\alpha_n(f)(\Lambda,\,\phi) := f(n\,\Lambda,\phi)\,, \quad \forall(\Lambda,\phi)\in\pi_n\,, \tag{1.54}$$

while $\alpha_n(f)(\Lambda,\,\phi)=0$ outside π_n. This semigroup action preserves the rational subalgebra $\mathcal{B}_\mathbb{Q}$ generated by the $e_{1,a}\,, a\in\mathbb{Q}/\mathbb{Z}\,,$ since one has

$$\alpha_n(e_{k,a}) = \pi_n\;e_{k,a/n}\,, \tag{1.55}$$

(independently of the choice of the solution $b=a/n$ of $n\,b=a$) and we shall now show that the projections π_n belong to $\mathcal{B}_\mathbb{Q}$.

Proof of Theorem 1.13

Using (1.53) one can express π_n as a rational linear combination of the $e_{k,a}$, with k even, but special care is needed when n is a power of two. The coefficient of $\gamma_k\,\pi_N$ in (1.53), when $N=p^b$ is a prime power, is given by $(2^k-2)(p-1)p^{b-1} - p^b(p^{k-1}-1)$, which does not vanish unless $p=2$, and is $-p^{b-1}(2-3p+p^2)$ for $k=2$. Thus, one can express π_N as a linear combination of the $e_{2,a}$ by induction on b. For $p=2$, $N=2^b$, $b>1$ the coefficient of $\gamma_k\,\pi_N$ in (1.53) is zero but the coefficient of $\gamma_k\,\pi_{N/2}$ is $-2^{b-2}(2^k-1)(2^k-2)\neq 0$ for k even. This allows one to express π_N as a linear combination of the $e_{2,a}$ by induction on b. Thus, for instance, π_2 is given by

$$\pi_2 = 3 + 2\sum_{4\,a=0} e_{2,a}.$$

In general, π_{2^n} involves $\displaystyle\sum_{2^{n+1}\,a=0} e_{2,a}$.

Since for relatively prime integers n, m one has $\pi_{nm}=\pi_n\,\pi_m$, we see that the algebra $\mathcal{B}_\mathbb{Q}$ generated over $\mathbb{Q}$ by the $e_{1,a}$ contains all the projections π_n. In order to show that $\mathcal{B}_\mathbb{Q}$ contains the $e(r)$ it is enough to show that for any prime power $N=p^b$ it contains $e(\frac{1}{N})$. This is proved by induction on b. Multiplying (1.53) by $1-\pi_p$ and using $(1-\pi_p)\,\pi_{p^l}=0$ for $l>0$ we get the equalities

$$(1-\pi_p)\sum_{N\,a=0} e_{k,a} = (N^k+2^k-2)\,\gamma_k\,(1-\pi_p)\,.$$

Let then $z(j) = (1-\pi_p)\,e_{1,\frac{j}{N}}$. The above relations together with(1.49) show that in the reduced algebra $(\mathcal{B}_\mathbb{Q})_{1-\pi_p}$ one has, for all k,

$$\sum_{j=1}^{N-1} P_k(z(j)) = (N^k-1)\,\gamma_k\,.$$

Thus, for $j \in \{1, ..., N-1\}$, the symmetric functions of the $z(j)$ are fixed rational numbers σ_h. In particular $z = z(1)$ fulfills

$$Q(z) = z^{N-1} + \sum_{1}^{N-1} (-1)^h \, \sigma_h \, z^{N-1-h} = 0$$

and $\pm \frac{1}{2}$ is not a root of this equation, whose roots are the $\dfrac{1}{2i} \cot(\dfrac{\pi j}{N})$. This allows us, using the companion matrix of Q, to express the Cayley transform of $2z$ as a polynomial with rational coefficients,

$$\frac{2z+1}{2z-1} = \sum_{0}^{N-2} \alpha_n \, z^n .$$

One then has

$$\sum_{0}^{N-2} \alpha_n \, z^n = (1 - \pi_p) \, e(\frac{1}{N}),$$

where the left-hand side belongs to $\mathcal{B}_{\mathbb{Q}}$ by construction. Now $\pi_p \, e(\frac{1}{N})$ is equal to $\alpha_p(e(\frac{p}{N}))$. It follows from the induction hypothesis on b, $(N = p^b)$, that $e(\frac{p}{N}) \in \mathcal{B}_{\mathbb{Q}}$ and therefore using (1.55) that $\alpha_p(e(\frac{p}{N})) \in \mathcal{B}_{\mathbb{Q}}$. Thus, we get $e(\frac{1}{N}) \in \mathcal{B}_{\mathbb{Q}}$ as required. This proves the first part. To get the second notice that the cross product by $\mathbb{N}^{\times}$ is obtained by adjoining to the rational group ring of $\mathbb{Q}/\mathbb{Z}$ the isometries μ_n and their adjoints μ_n^* with the relation

$$\mu_n \, f \, \mu_n^* = \alpha_n(f) , \quad \forall f \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] ,$$

which gives the rational algebra $\mathcal{A}_{\mathbb{Q}}$. $\Box$

It is not true, however, that the division relations (1.53) combined with the multiplicative relations (1.49) suffice to present the algebra. In particular there are more elaborate division relations which we did not need in the above proof. In order to formulate them, we let for $d|N$, $\pi(N, d)$ be the projection belonging to the algebra generated by the π_b, $b|N$, and corresponding to the subset

$$\gcd(N, \, (\Lambda, \phi)) = N/d$$

so that

$$\pi(N, d) = \pi_{N/d} \prod_{k|d} (1 - \pi_{k \, N/d}),$$

where the product is over non trivial divisors $k \neq 1$ of d.

Proposition 1.16 *The $e_{k,a}$, $a \in \mathbb{Q}/\mathbb{Z}$, k odd, fulfill for any $x \in \mathbb{Q}/\mathbb{Z}$ and any integer N the relation*

$$\frac{1}{N} \sum_{N \, a = 0} e_{k, x+a} = \sum_{d|N} \pi(N, d) \, d^{\, k-1} \, e_{k, d \, x} .$$

Proof. To prove this, let (Λ,ϕ) be such that $\gcd(N,(\Lambda,\phi)) = N/d = m$ and assume by homogeneity that $\Lambda = \mathbb{Z}$. Then when a ranges through the $\frac{j}{N}$, $j \in \{0,...,N-1\}$, the $\phi(a)$ range m-times through the $\frac{j}{d}$, $j \in \{0,...,d-1\}$. Thus the left-hand side of (1.16) gives m-times

$$c(\mathbb{Z})^k \sum_{j=0}^{d-1} \sum_{y\in\mathbb{Z}+\phi(x)+\frac{j}{d}} y^{-k} = c(\mathbb{Z})^k\, d^k \sum_{y\in\mathbb{Z}+\phi(d\,x)} y^{-k}.$$

This is clear when $y = 0$ does not appear in the sums involved. When it does one has, for $\epsilon \notin \frac{\mathbb{Z}}{d}$,

$$\sum_{j=0}^{d-1} \sum_{y\in\mathbb{Z}+\phi(x)+\frac{j}{d}} (y+\epsilon)^{-k} = d^k \sum_{y\in\mathbb{Z}+\phi(d\,x)} (y+\, d\,\epsilon)^{-k}.$$

Subtracting the pole part on both sides and equating the finite values gives the desired equality, since for odd k the value of $\epsilon_{k,a}(\Lambda,\phi)$ for $\phi(a) = 0$ can be written as the finite value of

$$\sum_{y\in\Lambda+\phi(a)} (y+\epsilon)^{-k}.$$

For even k this no longer holds and the finite value $\gamma_k c(\Lambda)^k$ is replaced by $(2^k-1)\,\gamma_k c(\Lambda)^k$. Thus when $\phi(d\,x) \in \mathbb{Z}$ one gets an additional term which is best taken care of by multiplying the right hand side in Proposition 1.16 by $(1-\pi_{\delta(d\,x)})$, with $\delta(y)$ the order of y in $\mathbb{Q}/\mathbb{Z}$, and adding corresponding terms to the formula, which becomes

$$\begin{aligned} \frac{1}{N} \sum_{N\,a=0} e_{k,x+a} &= \sum_{d|N} \pi(N,d)(1-\pi_{\delta(d\,x)})\, d^{\,k-1}\, e_{k,d\,x} \\ &+ \gamma_k \sum_{d|N} (d^{\,k-1} + \, d^{-1}(2^k-2))\,\pi(N,d)\,\pi_{\delta(d\,x)} \end{aligned} \tag{1.56}$$

These relations are more elaborate than the division relations for trigonometric functions. They restrict to the latter on the subset of invertible $\mathbb{Q}$-lattices, for which all π_n, $n \neq 1$ are zero and the only non-zero term in the r.h.s. is the term in $d = N$. The standard discussion of Eisenstein series in higher dimension is restricted to invertible $\mathbb{Q}$-lattices, but in our case the construction of the endomorphisms implemented by the μ_n requires the above extension to non-invertible $\mathbb{Q}$-lattices. We shall now proceed to do it in dimension 2.

1.7 The Determinant part of the GL_2-System

As we recalled in the previous sections, the algebra of the 1-dimensional system can be described as the semigroup cross product

$$C(R) \rtimes \mathbb{N}^{\times}.$$

Thus, one may wish to follow a similar approach for the 2-dimensional case, by replacing $C(R)$ by $C(M_2(R))$ and the semigroup action of $\mathbb{N}^\times$ by the semigroup action of $M_2(\mathbb{Z})^+$. Such construction can be carried out, as we discuss in this section, and it corresponds to the "determinant part" of the GL_2 system. It is useful to analyze what happens in this case first, before we discuss the full GL_2-system in the next section. In fact, this will show quite clearly where some important technical issues arise.
For instance, just as in the case of the BC algebra, where the time evolution acts on the isometries μ_n by n^{it} and leaves the elements of $C(R)$ fixed, the time evolution here is given by $\mathrm{Det}\,(m)^{it}$ on the isometries implementing the semigroup action of $m \in M_2(\mathbb{Z})^+$, while leaving $C(M_2(R))$ pointwise fixed. In this case, however, the vacuum state of the corresponding Hamiltonian is highly degenerate, because of the presence of the $\mathrm{SL}_2(\mathbb{Z})$ symmetry. This implies that the partition function and the KMS states below critical temperature can only be defined via the type II_1 trace Trace_Γ.
This issue will be taken care more naturally in the full GL_2-system, by first taking the classical quotient by $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on the space $M_2(R) \times \mathbb{H}$. This will resolve the degeneracy of the vacuum state and the counting of modes of the Hamiltonian will be on the coset classes $\Gamma\backslash M_2(\mathbb{Z})^+$.
The whole discussion of this section extends to $\mathrm{GL}(n)$ for arbitrary n and we shall briefly indicate how this is done, but we stick to $n = 2$ for definiteness.
We start with the action of the semigroup

$$M_2(\mathbb{Z})^+ = \{m \in M_2(\mathbb{Z}), \mathrm{Det}\,(m) > 0\} = \mathrm{GL}_2^+(\mathbb{Q}) \cap M_2(R) \tag{1.57}$$

on the compact space $M_2(R)$, given by left multiplication

$$\rho \mapsto \; m\,\rho, \tag{1.58}$$

where the product $m\rho$ takes place in $M_2(R)$ using the natural homomorphism

$$M_2(\mathbb{Z})^+ \to M_2(R), \tag{1.59}$$

which is the extension to two by two matrices of the inclusion of the ring $\mathbb{Z}$ in $\hat{\mathbb{Z}} = R$. The relevant C^*-algebra is the semi-group cross product

$$A = C(M_2(R)) \rtimes M_2(\mathbb{Z})^+. \tag{1.60}$$

It can be viewed as the C^*-algebra $C^*(G_2)$ of the étale groupoid G_2 of pairs (r, ρ), with $r \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(R)$ and $r\,\rho \in M_2(R)$, where the product takes place in $M_2(\mathbb{A}_f)$. The composition in G_2 is given by

$$(r_1, \rho_1) \circ (r_2, \rho_2) = \,(r_1\, r_2, \rho_2)\,, \quad \text{if } r_2\,\rho_2 = \,\rho_1$$

and the convolution of functions by

$$f_1 * f_2(r, \rho) := \sum f_1(rs^{-1}, s\,\rho)\, f_2(s, \rho)\,, \tag{1.61}$$

while the adjoint of f is

$$f^*(r,\rho) := \overline{f(r^{-1}, r\,\rho)} \tag{1.62}$$

(*cf.* the analogous expressions (1.39), (1.40), (1.41) in the 1-dimensional case).

A homomorphism $G_2 \to H$ of the groupoid G_2 to an abelian group H determines a *dual action* of the Pontrjagin dual of H on the algebra of G_2, as in the case of the time evolution σ_t, with $H = \mathbb{R}_+^*$ and its dual identified with $\mathbb{R}$. We shall use the same term "dual action" for nonabelian H.

The main structure is given by the *dual action* of $\mathrm{GL}_2^+(\mathbb{R})$ corresponding to the groupoid homomorphism j

$$j \,:\, G_2 \to \mathrm{GL}_2^+(\mathbb{R}), \quad j(r,\rho) = r \tag{1.63}$$

obtained from the inclusion $\mathrm{GL}_2^+(\mathbb{Q}) \subset \mathrm{GL}_2^+(\mathbb{R})$. As a derived piece of structure one gets the one parameter group of automorphisms $\sigma_t \in \mathrm{Aut}(A)$ which is dual to the determinant of the homomorphism j,

$$\sigma_t(f)(r,\rho) := \mathrm{Det}(r)^{it}\, f(r,\rho), \quad \forall f \in A\,. \tag{1.64}$$

The obtained C^*-dynamical system (A, σ_t) only involves $\mathrm{Det} \circ j$ and it does not fully correspond to the BC system. We shall make use of the full dual action of $\mathrm{GL}_2^+(\mathbb{R})$ later in the construction of the full GL_2 system.

The algebra $C(M_2(R))$ embeds as a $*$-subalgebra of A. The analogs of the isometries μ_n, $n \in \mathbb{N}^\times$ are the isometries μ_m, $m \in M_2(\mathbb{Z})^+$ given by

$$\mu_m(m,\rho) = 1\,, \quad \mu_m(r,\rho) = 0\,, \quad \forall r \neq m\,.$$

The range $\mu_m\,\mu_m^*$ of μ_m is the projection given by the characteristic function of the subset $P_m = m\,M_2(R) \subset M_2(R)$. It depends only on the lattice $L = m(\mathbb{Z}^2) \subset \mathbb{Z}^2$. Indeed, if $m, m' \in M_2(\mathbb{Z})^+$ fulfill $m(\mathbb{Z}^2) = m'(\mathbb{Z}^2)$, then $m' = m\gamma$ for some $\gamma \in \Gamma$, hence $m\,M_2(R) = m'\,M_2(R)$. Thus, we shall label this analog of the π_n by lattices

$$L \subset \mathbb{Z}^2 \mapsto \pi_L \in C(M_2(R)\,, \tag{1.65}$$

where π_L is the characteristic function of P_m, for any m such that $m(\mathbb{Z}^2) = L$. The algebra generated by the π_L is then governed by

$$\pi_L\,\pi_{L'} = \pi_{L\cap L'}\,, \qquad \pi_{\mathbb{Z}^2} = 1\,. \tag{1.66}$$

In fact, the complete rules are better expressed in terms of partial isometries $\mu(g, L)$, with $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $L \subset \mathbb{Z}^2$ a lattice, and $g(L) \subset \mathbb{Z}^2$, satisfying

$$\mu(g,L)(g,\rho) = \pi_L(\rho)\,, \quad \mu(g,L)(r,\rho) = 0\,, \quad \forall r \neq g\,.$$

One has

$$\mu(g_1,L_1)\,\mu(g_2,L_2) = \mu(g_1 g_2, g_2^{-1}(L_1) \cap L_2)\,, \tag{1.67}$$

and

$$\mu(g,L)^* = \mu(g^{-1}, g(L))\,. \tag{1.68}$$

The $\mu(g,L)$ generate the semi-group C^*-subalgebra $C^*(M_2(\mathbb{Z})^+) \subset A$ and together with $C(M_2(R))$ they generate A. The additional relations are

$$f\,\mu(g,L) = \mu(g,L)\,f^g\,, \quad \forall f \in C(M_2(R))\,, \quad g \in \mathrm{GL}_2^+(\mathbb{Q}), \tag{1.69}$$

where $f^g(y) := f(gy)$ whenever gy makes sense.

The action of $\mathrm{GL}_2(R)$ on $M_2(R)$ by right multiplication commutes with the semi-group action (1.58) of $M_2(\mathbb{Z})^+$ and with the time evolution σ_t. They define symmetries

$$\alpha_\theta \in \mathrm{Aut}(A,\sigma)\,.$$

Thus, we have a C^*-dynamical system with a compact group of symmetries. The following results show how to construct KMS_β-states for $\beta > 2$. We first describe a specific positive energy representation of the C^*-dynamical subsystem $(C^*(M_2(\mathbb{Z})^+),\,\sigma_t)$. We let $\mathcal{H} = \ell^2(M_2(\mathbb{Z})^+)$ with canonical basis ε_m, $m \in M_2(\mathbb{Z})^+$. We define $\pi(\mu(g,L))$ as the partial isometry in $\mathcal{H}$ with initial domain given by the span of

$$\varepsilon_m\,, \quad m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad (m_{11}, m_{21}) \in L\,,\ (m_{12}, m_{22}) \in L\,, \tag{1.70}$$

i.e. matrices m whose columns belong to the lattice $L \subset \mathbb{Z}^2$. On this domain we define the action of $\pi(\mu(g,L))$ by

$$\pi(\mu(g,L))\,\varepsilon_m = \varepsilon_{gm}\,. \tag{1.71}$$

Notice that the columns of gm belong to gL.

Proposition 1.17 *1) π is an involutive representation of $C^*(M_2(\mathbb{Z})^+)$ in $\mathcal{H}$.*
2) The Hamiltonian H given by $H\varepsilon_m = \log \mathrm{Det}(m)\,\varepsilon_m$ is positive and implements the time evolution σ_t:

$$\pi(\sigma_t(x)) = e^{itH}\,\pi(x)\,e^{-itH} \qquad \forall\, x \in C^*(M_2(\mathbb{Z})^+)\,.$$

3) $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on the right in $\mathcal{H}$ by

$$\rho(\gamma)\,\epsilon_m := \epsilon_{m\,\gamma^{-1}}\,, \quad \forall \gamma \in \Gamma\,, \quad m \in M_2(\mathbb{Z})^+\,.$$

and this action commutes with $\pi(C^*(M_2(\mathbb{Z})^+)$.

Proof. The map $m \mapsto gm$ is injective so that $\pi(\mu(g,L))$ is a partial isometry. Its range is the set of $h \in M_2(\mathbb{Z})^+$ of the form $g\,m$ where $\mathrm{Det}(m) > 0$ and the columns of m are in L. This means that $\mathrm{Det}(h) > 0$ and the columns of h are in $gL \subset \mathbb{Z}^2$. This shows that

$$\pi(\mu(g,L))^* = \pi(\mu(g^{-1}, gL)), \tag{1.72}$$

so that π is involutive on these elements.
Then the support of $\pi(\mu(g_1,L_1))\,\pi(\mu(g_2,L_2))$ is formed by the ε_m with columns of m in L_2, such that the columns of $g_2 m$ are in L_1. This is the same as the support of $\pi(\mu(g_1 g_2, g_2^{-1}L_1 \cap L_2))$ and the two partial isometries agree there. Thus, we get

$$\pi(\mu(g_1,L_1)\,\mu(g_2,L_2)) = \pi(\mu(g_1,L_1))\,\pi(\mu(g_2,L_2))\,. \tag{1.73}$$

Next, using (1.71) we see that

$$H\,\pi(\mu(g,L)) - \pi(\mu(g,L))\,H = \log(\mathrm{Det}\,g)\,\pi(\mu(g,L)), \tag{1.74}$$

since both sides vanish on the kernel while on the support one can use the multiplicativity of Det.
Now $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on the right in $\mathcal{H}$ by

$$\rho(\gamma)\,\epsilon_m := \epsilon_{m\,\gamma^{-1}}\,,\quad \forall\gamma\in\Gamma\,,\quad m\in M_2(\mathbb{Z})^+ \tag{1.75}$$

and this action commutes by construction with the algebra $\pi(C^*(M_2(\mathbb{Z})^+)$. □
The image $\rho(C^*(\Gamma))$ generates a type II_1 factor in $\mathcal{H}$, hence one can evaluate the corresponding trace Trace_Γ on any element of its commutant. We let

$$\varphi_\beta(x) := \mathrm{Trace}_\Gamma(\pi(x)\;e^{-\beta H})\,,\quad \forall x\in C^*(M_2(\mathbb{Z})^+) \tag{1.76}$$

and we define the normalization factor by

$$Z(\beta) = \mathrm{Trace}_\Gamma(e^{-\beta H})\,. \tag{1.77}$$

We then have the following:

Lemma 1.18 *1) The normalization factor $Z(\beta)$ is given by*

$$Z(\beta) = \zeta(\beta)\,\zeta(\beta-1),$$

where ζ is the Riemann ζ-function.
2) For all $\beta>2$, $Z^{-1}\,\varphi_\beta$ is a KMS$_\beta$ state on $C^*(M_2(\mathbb{Z})^+)$.

Proof. Any sublattice $L\subset\mathbb{Z}^2$ is uniquely of the form $L = \begin{bmatrix} a & b\\ 0 & d\end{bmatrix}\mathbb{Z}^2$, where $a,d\geq 1$, $0\leq b<d$ (*cf.* [50] p. 161). Thus, the type II_1 dimension of the action of Γ in the subspace of $\mathcal{H}$ spanned by the ε_m with $\mathrm{Det}\,m=N$ is the same as the cardinality of the quotient of $\{m\in M_2(\mathbb{Z})^+, \mathrm{Det}\,m=N\}$ by Γ acting on the right. This is equal to the cardinality of the set of matrices $\begin{bmatrix} a & b\\ 0 & d\end{bmatrix}$ as above with determinant $=N$. This gives $\sigma_1(N) = \sum\limits_{d|N} d$. Thus, $Z(\beta)$ is given by

$$\sum_{N=1}^{\infty}\frac{\sigma_1(N)}{N^\beta} = \zeta(\beta)\,\zeta(\beta-1)\,. \tag{1.78}$$

One checks the KMS$_\beta$-property of φ_β using the trace property of Trace_Γ together with the second equality in Proposition 1.17. □

Proposition 1.19 *1) For any $\theta \in \mathrm{GL}_2(R)$ the formula*

$$\pi_\theta(f)\,\epsilon_m := f(m\,\theta)\,\epsilon_m\,, \quad \forall m \in M_2(\mathbb{Z})^+$$

extends the representation π to an involutive representation π_θ of the cross product $A = C(M_2(R)) \rtimes M_2(\mathbb{Z})^+$ in $\mathcal{H}$.
2) Let $f \in C(M_2(\mathbb{Z}/N\mathbb{Z})) \subset C(M_2(R))$. Then $\pi_\theta(f) \in \rho(\Gamma_N)'$ where Γ_N is the congruence subgroup of level N.
3) For each $\beta > 2$ the formula

$$\psi_\beta(x) := \mathrm{Lim}_{\mathrm{N}\to\infty} \quad Z_N^{-1}\,\mathrm{Trace}_{\Gamma_N}(\pi_\theta(x)\,e^{-\beta H})\,, \quad \forall x \in A$$

defines a KMS$_\beta$ state on A, where $Z_N := \mathrm{Trace}_{\Gamma_N}(e^{-\beta H})$.

Proof. 1) The invertibility of θ shows that letting f_L be the characteristic function of P_L one has $\pi_\theta(f_L) = \pi_L$ independently of θ. Indeed $f_L(m\,\theta) = 1$ iff $m\,\theta \in P_L$ and this holds iff $m(\mathbb{Z}^2) \subset L$.
To check (1.69) one uses

$$f(gm\,\theta) = f^g(m\,\theta))\,, \quad \forall g \in \mathrm{GL}_2^+(\mathbb{Q})\,.$$

2) Let $p_N : M_2(R) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. It is a ring homomorphism. Let then $f = h \circ p_N$ where h is a function on $M_2(\mathbb{Z}/N\mathbb{Z})$. One has, for any $\gamma \in \Gamma_N$,

$$\begin{aligned} \pi_\theta(f)\,\rho(\gamma)\,\epsilon_m &= \pi_\theta(f)\,\epsilon_{m\,\gamma^{-1}} = \\ f(m\,\gamma^{-1}\theta)\,\epsilon_{m\,\gamma^{-1}} &= h(p_N(m\,\gamma^{-1}\theta))\,\epsilon_{m\,\gamma^{-1}}\,. \end{aligned}$$

The equality $p_N(\gamma) = 1$ shows that

$$p_N(m\,\gamma^{-1}\theta) = p_N(m)\,p_N(\gamma^{-1})\,p_N(\theta) = p_N(m)\,p_N(\theta) = p_N(m\,\theta),$$

hence

$$\pi_\theta(f)\,\rho(\gamma)\,\epsilon_m = \rho(\gamma)\,\pi_\theta(f)\,\epsilon_m\,.$$

3) One uses 2) to show that, for all N and $f \in C(M_2(\mathbb{Z}/N\mathbb{Z})) \subset C(M_2(R))$, the products $f\,\mu(g, L)$ belong to the commutant of $\rho(\Gamma_N)$. Since Γ_N has finite index in Γ it follows that for $\beta > 2$ one has $Z_N := \mathrm{Trace}_{\Gamma_N}(e^{-\beta H}) < \infty$. Thus, the limit defining $\psi_\beta(x)$ makes sense on a norm dense subalgebra of A and extends to a state on A by uniform continuity. One checks the KMS$_\beta$ condition on the dense subalgebra in the same way as above. □

It is not difficult to extend the above discussion to arbitrary n using ([51]). The normalization factor is then given by

$$Z(\beta) = \prod_0^{n-1} \zeta(\beta - k)\,.$$

What happens, however, is that the states ψ_β only depend on the determinant of θ. This shows that the above construction should be extended to involve not only the one-parameter group σ_t but in fact the whole dual action given by the groupoid homomorphism (1.63).

Definition 1.20 *Given a groupoid G and a homomorphism $j : G \to H$ to a group H, the "cross product" groupoid $G \times_j H$ is defined as the product $G \times H$ with units $G^{(0)} \times H$, range and source maps*

$$r(\gamma, \alpha) := (r(\gamma), j(\gamma)\, \alpha)\,, \quad s(\gamma, \alpha) := (s(\gamma), \alpha)$$

and composition

$$(\gamma_1, \alpha_1) \circ (\gamma_2, \alpha_2) := (\gamma_1 \circ \gamma_2, \alpha_2)\,.$$

In our case this cross product $\tilde{G_2} = G_2 \times_j \mathrm{GL}_2^+(\mathbb{R})$ corresponds to the groupoid of the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the locally compact space Z_0 of pairs $(\rho, \alpha) \in M_2(R) \times \mathrm{GL}_2^+(\mathbb{R})$ given by

$$g(\rho, \alpha) := (g\, \rho, g\, \alpha)\,, \quad \forall g \in \mathrm{GL}_2^+(\mathbb{Q})\,, \quad g\, \rho \in M_2(R)\,.$$

Since the subgroup $\Gamma \subset \mathrm{GL}_2^+(\mathbb{Q})$ acts freely and properly by translation on $\mathrm{GL}_2^+(\mathbb{R})$, one obtains a Morita equivalent groupoid S_2 by dividing $\tilde{G_2}$ by the following action of $\Gamma \times \Gamma$:

$$(\gamma_1, \gamma_2).(g, \rho, \alpha) := \,(\gamma_1\, g \gamma_2^{-1},\, \gamma_2 \rho, \gamma_2 \alpha). \tag{1.79}$$

The space of units $S_2^{(0)}$ is the quotient $\Gamma \backslash Z_0$. We let $p : Z_0 \to \Gamma \backslash Z_0$ be the quotient map. The range and source maps are given by

$$r(g, \rho, \alpha) := p(g\rho, g\alpha)\,, \quad s(g, \rho, \alpha) := p(\rho, \alpha)$$

and the composition is given by

$$(g_1, \rho_1, \alpha_1) \circ (g_2, \rho_2, \alpha_2) = (g_1 g_2, \rho_2, \alpha_2),$$

which passes to $\Gamma \times \Gamma$-orbits.

1.8 Commensurability of $\mathbb{Q}$-Lattices in $\mathbb{C}$ and the full GL_2-System

We shall now describe the full GL_2 C^*-dynamical system (A, σ_t). It is obtained from the system of the previous section by taking a cross product with the dual action of $\mathrm{GL}_2^+(\mathbb{R})$ *i.e.* from the groupoid S_2 that we just described. It admits an equivalent and more geometric description in terms of the notion of *commensurability* between $\mathbb{Q}$-lattices developed in section 1.6 above and we shall follow both points of view. The C^*-algebra A is a Hecke algebra, which is a variant of the *modular Hecke algebra* defined in ([10]). Recall from section 1.6:

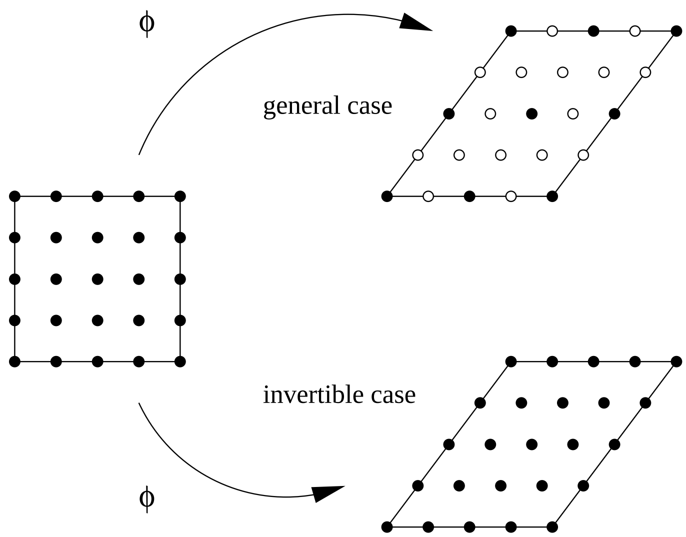

Figure 1.2: $\mathbb{Q}$-Lattices in $\mathbb{C}$.

Definition 1.21 *1) A $\mathbb{Q}$-lattice in $\mathbb{C}$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{C}$, and $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}\,\Lambda/\Lambda$ an homomorphism of abelian groups.*
2) Two $\mathbb{Q}$-lattices (Λ_j, ϕ_j) are commensurable iff Λ_j are commensurable and $\phi_1 - \phi_2 = 0$ modulo $\Lambda = \Lambda_1 + \Lambda_2$.

This is an equivalence relation $\mathcal{R}$ between $\mathbb{Q}$-lattices (Proposition 1.11). We use the basis $\{e_1 = 1, e_2 = -i\}$ of the $\mathbb{R}$-vector space $\mathbb{C}$ to let $\mathrm{GL}_2^+(\mathbb{R})$ act on $\mathbb{C}$ as $\mathbb{R}$-linear transformations. We let

$$\Lambda_0 := \mathbb{Z}\,e_1 + \mathbb{Z}\,e_2 = \mathbb{Z} + i\mathbb{Z}$$

Also we view $\rho \in M_2(R)$ as the homomorphism

$$\rho : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}.\Lambda_0/\Lambda_0\,, \quad \rho(a) = \rho_1(a)e_1 + \rho_2(a)e_2\,.$$

Proposition 1.22 *The map*

$$\gamma(g,\rho,\alpha) = ((\alpha^{-1}\,g^{-1}\Lambda_0,\, \alpha^{-1}\,\rho)\,, (\alpha^{-1}\,\Lambda_0,\, \alpha^{-1}\,\rho))\,, \quad \forall (g,\rho,\alpha) \in S_2$$

defines an isomorphism of locally compact étale groupoids between S_2 and the equivalence relation $\mathcal{R}$ of commensurability on the space of $\mathbb{Q}$-lattices in $\mathbb{C}$.

Proof. The proof is the same as for Proposition 1.12. □

We shall now describe the quotient of $S_2 \sim \mathcal{R}$ by the natural scaling action of $\mathbb{C}^*$. We view $\mathbb{C}^*$ as a subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ by the map

$$\lambda = a + i\,b \in \mathbb{C}^* \mapsto \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R}) \tag{1.80}$$

and identify the quotient $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{C}^*$ with $\mathbb{H}$ by the map

$$\alpha \in \mathrm{GL}_2^+(\mathbb{R}) \mapsto \tau = \alpha(i) \in \mathbb{H}. \tag{1.81}$$

Given a pair (Λ_j, ϕ_j) of commensurable $\mathbb{Q}$-lattices and a non zero complex number $\lambda \in \mathbb{C}^*$ the pair $(\lambda\Lambda_j, \lambda\phi_j)$ is still a pair of commensurable $\mathbb{Q}$-lattices. Moreover, one has

$$\gamma(g, \rho, \alpha\,\lambda^{-1}) = \lambda\,\gamma(g, \rho, \alpha)\,, \quad \forall \lambda \in \mathbb{C}^*\,. \tag{1.82}$$

The scaling action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices in $\mathbb{C}$ is not free, since the lattice Λ_0 for instance is invariant under multiplication by i. It follows that the quotient $S_2/\mathbb{C}^* \sim \mathcal{R}/\mathbb{C}^*$ is not a groupoid. One can nevertheless define its convolution algebra in a straightforward manner by restricting the convolution product on $S_2 \sim \mathcal{R}$ to functions which are homogeneous of *weight* 0, where weight k means

$$f(g, \rho, \alpha\,\lambda) = \lambda^k\, f(g, \rho, \alpha)\,, \quad \forall \lambda \in \mathbb{C}^*\,. \tag{1.83}$$

Let

$$Y = M_2(R) \times \mathbb{H}, \tag{1.84}$$

endowed with the natural action of $\mathrm{GL}_2^+(\mathbb{Q})$ by

$$\gamma \cdot (\rho, \tau) = \left(\gamma\,\rho,\ \frac{a\tau + b}{c\tau + d}\right), \tag{1.85}$$

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $(\rho, \tau) \in Y$. Let then

$$Z \subset \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \times_\Gamma Y \tag{1.86}$$

be the locally compact space quotient of $\{(g, y) \in \mathrm{GL}_2^+(\mathbb{Q}) \times Y\,,\ g\,y \in Y\}$ by the following action of $\Gamma \times \Gamma$:

$$(g, y) \mapsto (\gamma_1\, g\, \gamma_2^{-1}, \gamma_2\, y)\,, \quad \forall \gamma_j \in \Gamma\,.$$

The natural lift of the quotient map (1.81), together with proposition 1.22, first gives the identification of the quotient of Y by Γ with the space of $\mathbb{Q}$-lattices in $\mathbb{C}$ up to scaling, realized by the map

$$\theta : \Gamma\backslash Y \to (\text{Space of } \mathbb{Q}\text{-lattices in } \mathbb{C})/\mathbb{C}^* \ = \ X\,, \tag{1.87}$$

$$\theta(\rho, \tau) = (\Lambda, \phi)\,, \quad \Lambda = \mathbb{Z} + \mathbb{Z}\tau\,, \quad \phi(x) = \rho_1(x) - \tau\rho_2(x)\,.$$

It also gives the isomorphism $\theta : S_2/\mathbb{C}^* = Z \to \mathcal{R}/\mathbb{C}^*$,

$$\theta(g, y) = (\lambda\theta(gy), \theta(y)), \tag{1.88}$$

where $\lambda = \mathrm{Det}(g)^{-1}(c\tau + d)$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$ and $y = (\rho, \tau) \in Y$.

We let $\mathcal{A} = C_c(Z)$ be the space of continuous functions with compact support on Z. We view elements $f \in \mathcal{A}$ as functions on $\mathrm{GL}_2^+(\mathbb{Q}) \times Y$ such that

$$f(\gamma\, g, y) = f(g, y) \quad f(g\,\gamma, y) = f(g, \gamma\, y)\,, \quad \forall \gamma \in \Gamma\,,\ g \in \mathrm{GL}_2^+(\mathbb{Q})\,,\ y \in Y\,.$$

This does not imply that $f(g, y)$ only depends on the orbit $\Gamma.y$ but that it only depends on the orbit of y under the congruence subgroup $\Gamma \cap g^{-1}\Gamma\, g$.
We define the convolution product of two such functions by

$$(f_1 * f_2)(g, y) := \sum_{h \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, hy \in Y} f_1(gh^{-1}, hy)\; f_2(h, y) \tag{1.89}$$

and the adjoint by

$$f^*(g, y) := \overline{f(g^{-1}, g\,y)}. \tag{1.90}$$

Notice that these rules combine (1.23) and (1.61).

For any $x \in X$ we let $c(x)$ be the commensurability class of x. It is a countable subset of X and we want to define a natural representation in $l^2(c(x))$. We let p be the quotient map from Y to X. Let $y \in Y$ with $p(y) = x$ be an element in the preimage of x. Let

$$G_y = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) \,|\, gy \in Y\}\,.$$

The natural map $g \in G_y \mapsto p(gy) \in X$ is a surjection from $\Gamma\backslash G_y$ to $c(x)$ but it fails to be injective in degenerate cases such as $y = (0, \tau)$ with $\tau \in \mathbb{H}$ a complex multiplication point (*cf.* Lemma 1.28). This corresponds to the phenomenon of holonomy in the context of foliations ([11]). To handle it one defines the representation π_y directly in the Hilbert space $\mathcal{H}_y = l^2(\Gamma\backslash G_y)$ of left Γ-invariant functions on G_y by

$$(\pi_y(f)\,\xi)(g) := \sum_{h \in \Gamma\backslash G_y} f(gh^{-1}, hy)\,\xi(h)\,, \quad \forall g \in G_y\,, \tag{1.91}$$

for $f \in \mathcal{A}$ and $\xi \in \mathcal{H}_y$.

Proposition 1.23 *1) The vector space $\mathcal{A}$ endowed with the product $*$ and the adjoint $f \mapsto f^*$ is an involutive algebra.*

2) For any $y \in Y$, π_y defines a unitary representation of $\mathcal{A}$ in $\mathcal{H}_y$ whose unitary equivalence class only depends on $x = p(y)$.

3) The completion of $\mathcal{A}$ for the norm given by

$$||f|| := \mathrm{Sup}_{y \in Y}\, ||\pi_y(f)||$$

is a C^*-algebra.

The proof of (1) and (2) is similar to ([10], Proposition 2). Using the compactness of the support of f, one shows that the supremum is finite for any $f \in \mathcal{A}$ (cf. [11]). □

Remark 1.24 The locally compact space Z of (1.86) is not a groupoid, due to the torsion elements in Γ, which give nontrivial isotropy under scaling, for the square and equilateral lattices. Nonetheless, Proposition 1.23 yields a well defined C^*-algebra. This can be viewed as a subalgebra of the C^*-algebra of the groupoid obtained by replacing Γ by its commutator subgroup in the definition of S_2 as in (1.79).

We let σ_t be the one parameter group of automorphisms of A given by

$$\sigma_t(f)(g,y) = (\mathrm{Det}\, g)^{it}\, f(g,y)\,. \tag{1.92}$$

Notice that since X is not compact (but still locally compact) the C^*-algebra A does not have a unit, hence the discussion of Proposition 1.2 applies.

The one parameter group σ_{2t} (1.92) is the modular automorphism group associated to the regular representation of $\mathcal{A}$. To obtain the latter we endow $X = \Gamma\backslash Y$ with the measure

$$dy = \, d\rho \times d\mu(\tau),$$

where $d\rho = \prod d\rho_{ij}$ is the normalized Haar measure of the additive compact group $M_2(R)$ and $d\mu(\tau)$ is the Riemannian volume form in $\mathbb{H}$ for the Poincaré metric, normalized so that $\mu(\Gamma\backslash\mathbb{H}) = 1$. We then get the following result.

Proposition 1.25 *The expression*

$$\varphi(f) = \int_X f(1,y)\, dy\,. \tag{1.93}$$

defines a state on A, which is a KMS$_2$ state for the one parameter group σ_t.

Proof. At the measure theory level, the quotient $X = \Gamma\backslash Y$ is the total space over $\Gamma\backslash\mathbb{H}$ of a bundle with fiber the probability space $M_2(R)/\{\pm 1\}$, thus the total mass $\int_X \, dy \, = 1$. One gets

$$\varphi(f^* * f) = \int_X \sum_{h\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, hy\in Y} \overline{f(h,y)} f(h,y)\, dy\,, \quad \forall f \in \mathcal{A}\,,$$

which suffices to get the Hilbert space $\mathcal{H}$ of the regular representation and the cyclic vector ξ implementing the state φ, which corresponds to

$$\xi(g,y) = 0\,, \quad \forall g \notin \Gamma\,, \quad \xi(1,y) = 1\,, \quad \forall y \in X\,.$$

The measure $d\rho$ is the product of the additive Haar measures on column vectors, hence one gets

$$d(g\,\rho) = \,(\mathrm{Det}\, g)^{-2}\, d\rho\,, \quad \forall g \in \mathrm{GL}_2^+(\mathbb{Q})\,.$$

Let us prove that φ is a KMS$_2$ state. The above equality shows that, for any compactly supported continuous function α on $\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \times_\Gamma Y$, one has

$$\int_X \sum_{h\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \alpha(h,y)\, dy \, = \int_X \sum_{k\in\Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})} \alpha(k^{-1}, k\, y)\, (\mathrm{Det}\, k)^{-2}\, dy\,. \tag{1.94}$$

Let then $f_j \in \mathcal{A}$ and define $\alpha(h, y) = 0$ unless $hy \in Y$ while otherwise

$$\alpha(h, y) = f_1(h^{-1}, h\, y)\, f_2(h, y)(\mathrm{Det}\, h)^{it-2}.$$

The l.h.s. of (1.94) is then equal to $\varphi(f_1\, \sigma_z(f_2))$ for $z = t + 2i$. The r.h.s. of (1.94) gives $\varphi(\sigma_t(f_2)\, f_1)$ and (1.94) gives the desired equality $\varphi(f_1\, \sigma_{t+2i}(f_2)) = \varphi(\sigma_t(f_2)\, f_1)$. $\square$

We can now state the main result on the analysis of KMS states on the C^*-dynamical system (A, σ_t). Recall that a $\mathbb{Q}$-lattice $l = (\Lambda, \phi)$ is invertible if ϕ is an isomorphism (Definition 1.10). We have the following result.

Theorem 1.26 *1) For each invertible $\mathbb{Q}$-lattice $l = (\Lambda, \phi)$, the representation π_l is a positive energy representation of the C^*-dynamical system (A, σ_t).*
2) For $\beta > 2$ and $l = (\Lambda, \phi)$ an invertible $\mathbb{Q}$-lattice, the formula

$$\varphi_{\beta,l}(f) = Z^{-1} \sum_{\Gamma \backslash M_2(\mathbb{Z})^+} f(1, m\, \rho, m(\tau))\, \mathrm{Det}(m)^{-\beta}\,,$$

defines an extremal KMS$_\beta$ state $\varphi_{\beta,l}$ on (A, σ_t), where $Z = \zeta(\beta)\, \zeta(\beta - 1)$ is the partition function.
3) For $\beta > 2$ the map $l \mapsto \varphi_{\beta,l}$ is a bijection from the space of invertible $\mathbb{Q}$-lattices (up to scaling) to the space $\mathcal{E}_\beta$ of extremal KMS$_\beta$ states on (A, σ_t).

The proof of 1) reflects the following fact, which in essence shows that the invertible $\mathbb{Q}$-lattices are *ground states* for our system.

Lemma 1.27 *1) Let $s : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}^2$ be a section of the projection $\pi : \mathbb{Q}^2 \to \mathbb{Q}^2/\mathbb{Z}^2$. Then the set of $s(a + b) - s(a) - s(b)\,,\ a, b \in \mathbb{Q}^2/\mathbb{Z}^2$, generates $\mathbb{Z}^2$.*
2) Let $l = (\Lambda, \phi)$ be an invertible $\mathbb{Q}$-lattice and $l' = (\Lambda', \phi')$ be commensurable with l. Then $\Lambda \subset \Lambda'$.

Proof. 1) Let $L \subset \mathbb{Z}^2$ be the subgroup generated by the $s(a + b) - s(a) - s(b)\,,\ a, b \in \mathbb{Q}^2/\mathbb{Z}^2$. If $L \neq \mathbb{Z}^2$ we can assume, after a change of basis, that for some prime number p one has $L \subset p\,\mathbb{Z} \oplus \mathbb{Z}$. Restricting s to the p-torsion elements of $\mathbb{Q}^2/\mathbb{Z}^2$ and multiplying it by p, we get a morphism of groups

$$\mathbb{Z}/p\,\mathbb{Z} \oplus \mathbb{Z}/p\,\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\,\mathbb{Z},$$

which is a section of the projection

$$\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\,\mathbb{Z} \to \mathbb{Z}/p\,\mathbb{Z} \oplus \mathbb{Z}/p\,\mathbb{Z}.$$

This gives a contradiction, since the group $\mathbb{Z}/p^2\mathbb{Z} \oplus \mathbb{Z}/p\,\mathbb{Z}$ contains elements of order p^2.
2) Let s (resp. s') be a lift of ϕ modulo Λ (resp. of ϕ' modulo Λ'). Since $\phi - \phi' = 0$ modulo $\Lambda'' = \Lambda + \Lambda'$ one has $s(a) - s'(a) \in \Lambda + \Lambda'$ for all $a \in \mathbb{Q}^2/\mathbb{Z}^2$.

This allows one to correct s modulo Λ and s' modulo Λ' so that $s = s'$. Then for any $a, b \in \mathbb{Q}^2/\mathbb{Z}^2$ one has $s(a+b) - s(a) - s(b) \in \Lambda \cap \Lambda'$ and the first part of the lemma together with the invertibility of ϕ show that $\Lambda \cap \Lambda' = \Lambda$. $\Box$

Given $y \in Y$ we let H_y be the diagonal operator in $\mathcal{H}_y$ given by

$$(H_y\,\xi)(h) := \log(\mathrm{Det}(h))\ \xi(h)\,, \quad \forall h \in G_y \tag{1.95}$$

It implements the one parameter group σ_t *i.e.*

$$\pi_y(\sigma_t(x)) = \, e^{itH_y}\,\pi_y(x)\,e^{-itH_y}\,, \quad \forall x \in A\,. \tag{1.96}$$

In general the operator H_y is not positive but when the lattice $l = (\Lambda, \phi) = \theta(p(y))$ is invertible one has

$$\mathrm{Det}(h) \in \mathbb{N}^*\,, \quad \forall h \in G_y,$$

hence $H_y \geq 0$. This proves the first part of the theorem. The basis of the Hilbert space $\mathcal{H}_y$ is then labeled by the lattices Λ' containing Λ and the operator H_y is diagonal with eigenvalues the logarithms of the orders $\Lambda' : \Lambda$. Equivalently, one can label the orthonormal basis ϵ_m by the coset space $\Gamma\backslash M_2(\mathbb{Z})^+$. Thus, the same counting as in the previous section (cf.[51]) shows that

$$Z = \,\mathrm{Trace}(e^{-\beta H_y}) = \zeta(\beta)\,\zeta(\beta - 1)$$

and in particular that it is finite for $\beta > 2$. The KMS_β property of the functional

$$\varphi_{\beta,l}(f) = Z^{-1}\,\mathrm{Trace}(\pi_y(f)\ e^{-\beta H_y})$$

then follows from (1.96). One has, using (1.91) for $y = (\rho, \tau) \in Y$,

$$\langle \pi_y(f)(\epsilon_m),\ \epsilon_m \rangle \ = \ f(1, m\,\rho, m(\tau))\,,$$

hence we get the following formula for $\varphi_{\beta,l}$:

$$\varphi_{\beta,l}(f) = Z^{-1} \sum_{\Gamma\backslash M_2(\mathbb{Z})^+} f(1, m\,\rho, m(\tau))\,\mathrm{Det}(m)^{-\beta}\,. \tag{1.97}$$

Finally, the irreducibility of the representation π_y follows as in [11] p.562 using the absence of holonomy for invertible $\mathbb{Q}$-lattices. This completes the proof of 2) of Theorem 1.26.

In order to prove 3) of Theorem 1.26 we shall proceed in two steps. The first shows (Proposition 1.30 below) that KMS_β states are given by measures on the space X of $\mathbb{Q}$-lattices (up to scaling). The second shows that when $\beta > 2$ this measure is carried by the commensurability classes of invertible $\mathbb{Q}$-lattices.

We first describe the stabilizers of the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the space of $\mathbb{Q}$-lattices in $\mathbb{C}$.

Lemma 1.28 *Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $g \neq 1$ and $y \in Y$, $y = (\rho, \tau)$ such that $gy = y$. Then $\rho = 0$. Moreover $g \in \mathbb{Q}^* \subset \mathrm{GL}_2^+(\mathbb{Q})$ unless τ is an imaginary quadratic number in which case $g \in K^* \subset \mathrm{GL}_2^+(\mathbb{Q})$ where $K = \mathbb{Q}(\tau)$ is the corresponding quadratic field.*

Proof. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $g(\tau) = \tau$ means $a\tau + b = \tau\,(c\tau + d)$. If $c \neq 0$ this shows that τ is an imaginary quadratic number. Let $K = \mathbb{Q}(\tau)$ be the corresponding field and let $\{\tau, 1\}$ be the natural basis of K over $\mathbb{Q}$. Then the multiplication by $(c\tau + d)$ is given by the transpose of the matrix g. Since $g \neq 1$ and K is a field we get

$$g - 1 = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Q})$$

and thus $(g-1)\rho = 0$ implies $\rho = 0$. If $c = 0$ then $a\tau + b = \tau\,(c\tau + d)$ implies $a = d$, $b = 0$ so that $g \in \mathbb{Q}^* \subset \mathrm{GL}_2^+(\mathbb{Q})$. Since $g \neq 1$ one gets $\rho = 0$. $\Box$

We let $X = \Gamma \backslash Y$ be the quotient and $p : Y \to X$ the quotient map. We let F be the closed subset of X, $F = p(\{(0, \tau); \tau \in \mathbb{H}\})$.

Lemma 1.29 *Let $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $g \notin \Gamma$ and $x \in X$, $x \notin F$. There exists a neighborhood V of x, such that*

$$p(g\,p^{-1}(V)) \cap V = \emptyset$$

Proof. Let $\Gamma_0 = \Gamma \cap g^{-1}\Gamma g$ and $X_0 = \Gamma_0 \backslash Y$. For $x_0 \in X_0$ the projections $p_1(x_0) = p(y)$ and $p_2(x_0) = p(gy)$ are independent of the representative $y \in Y$. Moreover if $p_1(x_0) \notin F$ then $p_1(x_0) \neq p_2(x_0)$ by Lemma 1.28. By construction Γ_0 is of finite index n in Γ and the fiber $p_1^{-1}(x)$ has at most n elements. Let then $W \subset X_0$ be a compact neighborhood of $p_1^{-1}(x)$ in X_0 such that $x \notin p_2(W)$. For $z \in V \subset X$ sufficiently close to x one has $p_1^{-1}(z) \subset W$ and thus $p_2(p_1^{-1}(z)) \subset V^c$ which gives the result. $\Box$

We can now prove the following.

Proposition 1.30 *Let $\beta > 0$ and φ a KMS$_\beta$ state on (A, σ_t). Then there exists a probability measure μ on $X = \Gamma \backslash Y$ such that*

$$\varphi(f) = \int_X f(1, x)\, d\mu(x)\,, \quad \forall f \in A\,.$$

Proof. Let $g_0 \in \mathrm{GL}_2^+(\mathbb{Q})$ and $f \in C_c(Z)$ such that

$$f(g, y) = 0\,, \quad \forall g \notin \Gamma\, g_0\, \Gamma\,.$$

Since any element of $C_c(Z)$ is a finite linear combination of such functions, it is enough to show that $\varphi(f) = 0$ provided $g_0 \notin \Gamma$. Let $h_n \in C_c(X)$, $0 \leq h_n \leq 1$

with support disjoint from F and converging pointwise to 1 in the complement of F. Let $u_n \in A$ be given by

$$u_n(1,y) := h_n(y)\,, \quad u_n(g,y) = 0\,, \quad \forall g \notin \Gamma\,.$$

The formula

$$\Phi(f)(g,\tau) := f(g,0,\tau) \quad \forall f \in A \tag{1.98}$$

defines a homomorphism of (A, σ_t) to the C^* dynamical system (B, σ_t) obtained by specialization to $\rho = 0$, with convolution product

$$f_1 * f_2(\rho,\tau) = \sum_h f_1(gh^{-1}, h(\tau)) f_2(h,\tau),$$

where now we have no restriction on the summation, as in [10].

For each $n \in \mathbb{N}^*$ we let

$$\mu_{[n]}(g,y) = 1 \text{ if } g \in \Gamma.[n]\,, \quad \mu_{[n]}(g,y) = 0 \text{ if } g \notin \Gamma.[n]\,. \tag{1.99}$$

One has $\mu_{[n]}^*\,\mu_{[n]} = 1$ and $\sigma_t(\mu_{[n]}) = n^{2it}\,\mu_{[n]}\,, \quad \forall t \in \mathbb{R}$. Moreover, the range $\pi(n) = \mu_{[n]}\,\mu_{[n]}^*$ of $\mu_{[n]}$ is the characteristic function of the set of $\mathbb{Q}$-lattices that are divisible by n, *i.e.* those of the form $(\Lambda, n\,\phi)$.

Let $\nu_{[n]} = \Phi(\mu_{[n]})$. These are unitary multipliers of B. Since they are eigenvectors for σ_t, the system (B, σ_t) has no non-zero KMS_β positive functional. This shows that the pushforward of φ by Φ vanishes and by Proposition 1.5 that, with the notation introduced above,

$$\varphi(f) = \lim_n \varphi(f * u_n)\,.$$

Thus, since $(f * u_n)(g,y) = f(g,y)\,h_n(y)$, we can assume that $f(g,y) = 0$ unless $p(y) \in K$, where $K \subset X$ is a compact subset disjoint from F. Let $x \in K$ and V as in lemma 1.29 and let $h \in C_c(V)$. Then, upon applying the KMS_β condition (1.6) to the pair a, b with $a = f$ and

$$b(1,y) := h(y)\,, \quad b(g,y) = 0\,, \quad \forall g \notin \Gamma\,,$$

one gets $\varphi(b * f) = \varphi(f * b)$. One has $(b * f)(g,y) = h(gy)\,f(g,y)$. Applying this to $f * b$ instead of f and using $h(gy)\,h(y) = 0\,, \quad \forall y \in X$ we get $\varphi(f * b^2) = 0$ and $\varphi(f) = 0$, using a partition of unity on K. $\Box$

We let Det be the continuous map from $M_2(R)$ to R given by the determinant

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) \mapsto a\,d - b\,c \in R\,.$$

For each $n \in \mathbb{N}^*$, the composition $\pi_n \circ \mathrm{Det}$ defines a projection $\pi'(n)$, which is the characteristic function of the set of $\mathbb{Q}$-lattices whose determinant is divisible by n. If a $\mathbb{Q}$-lattice is divisible by n its determinant is divisible by n^2 and one

controls divisibility using the following family of projections $\pi_p(k,l)$. Given a prime p and a pair (k,l) of integers $k \leq l$, we let

$$\pi_p(k,l) := (\pi(p^k) - \pi(p^{k+1}))\,(\pi'(p^{k+l}) - \pi'(p^{k+l+1})). \tag{1.100}$$

This corresponds, when working modulo $N = p^b, b > l$, to matrices in the double class of

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad v_p(a) = k\,, \quad v_p(d) = l\,,$$

where v_p is the p-adic valuation.

Lemma 1.31

- *Let φ be a KMS$_\beta$ state on (A, σ_t). Then, for any prime p and pair (k,l) of integers $k < l$, one has*

$$\varphi(\pi_p(k,l)) = p^{-(k+l)\beta}\, p^{l-k}\,(1+p^{-1})\,(1-p^{-\beta})\,(1-p^{1-\beta})$$

while for $k = l$ one has

$$\varphi(\pi_p(l,l)) = p^{-2l\beta}\,(1-p^{-\beta})\,(1-p^{1-\beta})\,.$$

- *For distinct primes p_j one has*

$$\varphi(\prod \pi_{p_j}(k_j,l_j)) = \prod \varphi(\pi_{p_j}(k_j,l_j)).$$

Proof. For each $n \in \mathbb{N}^*$ we let $\nu_n \in M(A)$ be given by

$$\nu_n(g,y) = 1\,, \quad \forall g \in \Gamma \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} \Gamma\,, \quad \nu_n(g,y) = 0 \quad \text{otherwise}\,.$$

One has $\sigma_t(\nu_n) = n^{it}\,\nu_n\,, \quad \forall t \in \mathbb{R}$. The double class $\Gamma \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} \Gamma$ is the union of the left Γ-cosets of the matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where $a\,d = n$ and $\gcd(a,b,d) = 1$. The number of these left cosets is

$$\omega(n) := n \prod_{p\,\text{prime},\, p|n} (1+p^{-1})$$

and

$$\nu_n^* * \nu_n(1,y) = \omega(n)\,, \quad \forall y \in Y. \tag{1.101}$$

One has

$$\nu_n * \nu_n^*(1,y) = \sum_{h \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, hy \in Y} \nu_n(h^{-1}, hy)^2\,.$$

With $y = (\rho, \tau)$, the r.h.s. is independent of τ and only depends upon the $\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}) - \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ double class of $\rho_n = p_n(\rho) \in M_2(\mathbb{Z}/n\mathbb{Z})$.

Let us assume that $n = p^l$ is a prime power. We can assume that $\rho_n = p_n(\rho)$ is of the form

$$\rho_n = \begin{bmatrix} p^a & 0 \\ 0 & p^b \end{bmatrix}, \quad 0 \le a \le b \le l\,.$$

We need to count the number $\omega(a,b)$ of left Γ-cosets $\Gamma\, h_j$ in the double class $\Gamma \begin{bmatrix} n^{-1} & 0 \\ 0 & 1 \end{bmatrix} \Gamma$ such that $h_j y \in Y$ *i.e.* $h_j \rho \in M_2(R)$. A full set of representatives of the double class is given by $h_j = (\alpha_j^t)^{-1}$ where the α_j are

$$\alpha_0 = \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} \qquad \alpha(s) = \begin{bmatrix} 1 & s \\ 0 & n \end{bmatrix}, s \in \{0, 1, ..., n-1\}$$

and for $x \in \{1, 2, ..., l-1\}$, $s \in \mathbb{Z}/p^{l-x}\mathbb{Z}$ prime to p

$$\alpha(x,s) = \begin{bmatrix} p^x & s \\ 0 & p^{l-x} \end{bmatrix}.$$

The counting gives

- $\omega(a,b) = 0$ if $b < l$.
- $\omega(a,b) = p^a$ if $a < l, b \ge l$.
- $\omega(a,b) = p^l(1 + p^{-1})$ if $a \ge l$.

Let $e_p(i,j)$, $(i \le j)$ be the projection corresponding to $a \ge i$, $b \ge j$. Then for $i < j$ one has

$$\pi_p(i,j) = e_p(i,j) - e_p(i+1,j) - e_p(i,j+1) + e_p(i+1,j+1) \tag{1.102}$$

while

$$\pi_p(j,j) = e_p(j,j) - e_p(j,j+1). \tag{1.103}$$

The computation above gives

$$\nu_n * \nu_n^*(1,y) = p^l(1+p^{-1})e_p(l,l) + \sum_0^{l-1} p^k \left(e_p(k,l) - e_p(k+1,l)\right), \tag{1.104}$$

where we omit the variable y in the r.h.s.
Let φ be a KMS$_\beta$ state, and $\sigma(k,l) := \varphi(e_p(k,l))$. Then, applying the KMS$_\beta$ condition to the pair $(\mu_{[p]}f, \mu_{[p]}^*)$ for $f \in C(X)$, one gets

$$\sigma(k,l) = p^{-2k\beta}\,\sigma(0, l-k)\,.$$

Let $\sigma(k) = \sigma(0,k)$. Upon applying the KMS$_\beta$ condition to (ν_n, ν_n^*), one gets

$$p^l(1+p^{-1})p^{-l\beta} = p^l(1+p^{-1})p^{-2l\beta} + \sum_0^{l-1} p^k \left(p^{-2k\beta}\sigma(l-k) - p^{-2(k+1)\beta}\sigma(l-k-1)\right.\,.$$

Since $\sigma(0)=1$, this determines the $\sigma(n)$ by induction on n and gives

$$\sigma(n)=a\,p^{n(1-\beta)}+(1-a)\,p^{-2n\beta},$$

with

$$a=(1+p)\frac{p^{\beta}-1}{p^{1+\beta}-1}\,.$$

Combined with (1.102) and (1.103), this gives the required formulas for $\varphi(\pi_p(k,l))$ and the first part of the lemma follows.

To get the second part, one proceeds by induction on the number m of primes p_j. The function $f=\prod_1^{m-1}\pi_{p_j}(k_j,l_j)$ fulfills

$$f(h\,y)=f(y)\,,\quad \forall y\in Y\,,\quad \forall h\in\Gamma\begin{bmatrix}n^{-1}&0\\0&1\end{bmatrix}\Gamma,$$

where $n=p_m^l$. Thus, when applying the KMS_β condition to $(\nu_n\,f,\nu_n^*)$, the above computation applies with no change to give the result. $\Box$

Let us now complete the proof of 3) of Theorem 1.26. Let φ be a KMS_β state. Proposition 1.30 shows that there is a probability measure μ on X such that

$$\varphi(f)=\int_X f(1,x)\,d\mu(x)\,,\quad \forall f\in A\,.$$

With $y=(\rho,\tau)\in X$, Lemma 1.31 shows that the probability $\varphi(e_p(1,1))=\sigma(1,1)$ that a prime p divides ρ is $p^{-2\beta}$. Since the series $\sum p^{-2\beta}$ converges ($\beta>\frac{1}{2}$ would suffice here), it follows (*cf.* [44] Thm. 1.41) that, for almost all $y\in X$, ρ is only divisible by a finite number of primes. Next, again by Lemma 1.31, the probability that the determinant of ρ is divisible by p is

$$\varphi(e_p(0,1))=\sigma(1)=(1+p)p^{-\beta}-\,p^{1-2\beta}.$$

For $\beta>2$ the corresponding series $\sum\left((1+p)p^{-\beta}-\,p^{1-2\beta}\right)$ is convergent. Thus, we conclude that with probability one

$$\rho_p\in \mathrm{GL}_2(\mathbb{Z}_p)\,,\quad \text{for almost all } p.$$

Moreover, since $\sum\varphi(\pi_p(k,l))=1$, one gets with probability one

$$\rho_p\in \mathrm{GL}_2(\mathbb{Q}_p)\,,\quad \forall p\,.$$

In other words, the measure μ gives measure one to finite idèles. (Notice that finite idèles form a Borel subset which is not closed.) However, when ρ is a finite idèle the corresponding $\mathbb{Q}$-lattice is commensurable to a unique invertible $\mathbb{Q}$-lattice. Then the KMS_β condition shows that the measure μ is entirely determined by its restriction to invertible $\mathbb{Q}$-lattices, so that, for some probability measure ν,

$$\varphi=\int\,\varphi_{\beta,l}\;d\nu(l).$$

It follows that the Choquet simplex of extremal KMS_β states is the space of probability measures on the locally compact space

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$$

of invertible $\mathbb{Q}$-lattices [3] and its extreme points are the $\varphi_{\beta,l}$. $\Box$

In fact Lemma 1.31 admits the following corollary:

Corollary 1.32 *For $\beta \leq 1$ there is no KMS_β state on (A, σ_t).*

Proof. Indeed the value of $\varphi(\pi_p(k,l))$ provided by the lemma is strictly negative for $\beta < 1$ and vanishes for $\beta = 1$. In the latter case this shows that the measure μ is supported by $\{0\} \times \mathbb{H} \subset Y$ and one checks that no such measure fulfills the KMS condition for $\beta = 1$. $\Box$

In fact the measure provided by Lemma 1.31 allows us to construct a specific KMS_β state on (A, σ_t) for $1 < \beta \leq 2$. We shall analyze this range of values in Chapter III in connection with the renormalization group.

To get some feeling about what happens when $\beta \to 2$ from above, we shall show that, on functions f which are independent of τ, the states $\varphi_{\beta,l}$ converge weakly to the KMS_2 state φ of (1.93), independently of the choice of the invertible $\mathbb{Q}$-lattice l. Namely, we have

$$\varphi_{\beta,l}(f) \to \int_{M_2(R)} f(a)\, da\,.$$

Using the density of functions of the form $f \circ p_N$ among left Γ-invariant continuous functions on $M_2(R)$, this follows from:

Lemma 1.33 *For $N \in \mathbb{N}$, let $\Gamma(N)$ be the congruence subgroup of level N and*

$$Z_\beta = \sum_{\Gamma(N)\backslash M_2(\mathbb{Z})^+} \mathrm{Det}(m)^{-\beta}\,.$$

When $\beta \to 2$ one has, for any function f on $M_2(\mathbb{Z}/N\mathbb{Z})$,

$$Z_\beta^{-1} \sum_{\Gamma(N)\backslash M_2(\mathbb{Z})^+} f(p_N(m))\,\mathrm{Det}(m)^{-\beta} \to N^{-4} \sum_{M_2(\mathbb{Z}/N\mathbb{Z})} f(a)\,.$$

Proof. For $x \in M_2(\mathbb{Z}/N\mathbb{Z})$ we let

$$h(x) = \lim_{\beta\to 2} Z_\beta^{-1} \sum_{m\in\Gamma(N)\backslash M_2(\mathbb{Z})^+,\, p_N(m)=x} \mathrm{Det}(m)^{-\beta}$$

be the limit of the above expression, with f the characteristic function of the subset $\{x\} \subset M_2(\mathbb{Z}/N\mathbb{Z})$. We want to show that

$$h(x) = N^{-4}\,, \quad \forall x \in M_2(\mathbb{Z}/N\mathbb{Z})\,. \tag{1.105}$$

[3] *cf. e.g.* [39] for the standard identification of the set of invertible $\mathbb{Q}$-lattices with the above double quotient

Since p_N is a surjection $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ and $\Gamma(N)$ a normal subgroup of Γ, one gets

$$h(\gamma_1\, x \gamma_2) = h(x)\,, \quad \forall \gamma_j \in SL_2(\mathbb{Z}/N\mathbb{Z})\,. \tag{1.106}$$

Thus, to prove (1.105) we can assume that x is a diagonal matrix

$$x = \begin{bmatrix} n & 0 \\ 0 & n\,\ell \end{bmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z})\,.$$

Dividing both n and N by their g.c.d. k does not affect the validity of (1.105), since all $m \in \Gamma(N)\backslash M_2(\mathbb{Z})^+$ with $p_N(m) = x$ are of the form $k\,m'$, while $\mathrm{Det}(m)^{-\beta} = k^{-2\beta}\,\mathrm{Det}(m')^{-\beta}$. This shows that (1.105) holds for $n = 0$ and allows us to assume that n is coprime to N. Let then r be the g.c.d. of ℓ and N. One can then assume that $x = \begin{bmatrix} n & 0 \\ 0 & n'\,r \end{bmatrix}$, with $r|N$ and with n and n' coprime to N. Let $\Delta \subset SL_2(\mathbb{Z}/N\mathbb{Z})$ be the diagonal subgroup. The left coset $\Delta\, x \subset M_2(\mathbb{Z}/N\mathbb{Z})$ only depends on r and the residue $\delta \in (\mathbb{Z}/N'\mathbb{Z})^*$ of $n\,n'$ modulo $N' = N/r$. It is the set of all diagonal matrices of the form

$$y = \begin{bmatrix} n_1 & 0 \\ 0 & n_2\,r \end{bmatrix}\,, \quad n_1 \in (\mathbb{Z}/N\mathbb{Z})^*\,, \quad n_1\,n_2 = \delta\ (N')\,.$$

Let $\Gamma_\Delta(N) \subset \Gamma$ be the inverse image of Δ by p_N. By (1.106) h is constant on $\Delta\, x$, hence

$$h(x) = \lim_{\beta\to 2} Z_\beta^{-1} \sum_{m\in\Gamma_\Delta(N)\backslash M_2(\mathbb{Z})^+,\, p_N(m)\in\Delta\, x} \mathrm{Det}(m)^{-\beta}\,. \tag{1.107}$$

In each left coset $m \in \Gamma_\Delta(N)\backslash M_2(\mathbb{Z})^+$ with $p_N(m) \in \Delta\, x$ one can find a unique triangular matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a > 0$ coprime to N, $d > 0$ divisible by r, $a\,d/r = \delta\,(N')$ and $b = N\,b'$ with $0 \le b' < d$. Thus, we can rewrite (1.107) as

$$h(x) = \lim_{\beta\to 2} Z_\beta^{-1} \sum_{Y} d\,(ad)^{-\beta}, \tag{1.108}$$

where Y is the set of pairs of positive integers (a, d) such that

$$p_N(a) \in (\mathbb{Z}/N\mathbb{Z})^*\,, \quad r|d\,, \quad a\,d = r\,\delta\,(N).$$

To prove (1.105) we assume first that $r < N$ and write $N = N_1\,N_2$, where N_1 is coprime to N' and N_2 has the same prime factors as N'. One has $r = N_1\,r_2$, with $r_2|N_2$. An element of $\mathbb{Z}/N_2\mathbb{Z}$ is invertible iff its image in $\mathbb{Z}/N'\mathbb{Z}$ is invertible. To prove (1.105) it is enough to show that, for any of the r_2 lifts $\delta_2 \in \mathbb{Z}/N_2\mathbb{Z}$ of δ, one has

$$\lim_{\beta\to 2} Z_\beta^{-1} \sum_{Y'} d\,(ad)^{-\beta} = N_1\,N^{-4}, \tag{1.109}$$

where Y' is the set of pairs of positive integers (a, d) such that

$$p_N(a) \in (\mathbb{Z}/N\mathbb{Z})^*\,, \quad p_{N_2}(a\,d) = \delta_2.$$

We let 1_N be the trivial Dirichlet character modulo N. Then when $\mathcal{X}_{N_2}$ varies among Dirichlet characters modulo N_2 one has

$$\sum_{Y'} d\,(ad)^{-\beta} = \varphi(N_2)^{-1} \sum \mathcal{X}_{N_2}(\delta_2)^{-1}\, L(1_{N_1} \times \mathcal{X}_{N_2}, \beta)\, L(\mathcal{X}_{N_2}, \beta - 1),$$

where φ is the Euler totient function. Only the trivial character $\mathcal{X}_{N_2} = 1_{N_2}$ contributes to the limit (1.109), since the other L-functions are regular at 1. Moreover, the residue of $L(1_{N_2}, \beta - 1)$ at $\beta = 2$ is equal to $\frac{\varphi(N_2)}{N_2}$ so that, when $\beta \to 2$, we have

$$\sum_{Y'} d\,(ad)^{-\beta} \sim N_2^{-1}\, L(1_N, 2)\,(\beta - 2)^{-1}\,.$$

By construction one has

$$Z_\beta \sim |\Gamma : \Gamma(N)|\, \zeta(2)\,(\beta - 2)^{-1}\,,$$

where the order of the quotient group $\Gamma : \Gamma(N)$ is $N^3 \prod_{p|N} (1 - p^{-2})$ ([51]). Since

$$L(1_N, s) = \prod_{p|N} (1 - p^{-s})\, \zeta(s)$$

one gets (1.109). A similar argument handles the case $r = N$. $\Box$

The states $\varphi_{\beta,l}$ converge when $\beta \to \infty$ and their limits restrict to $C_c(X) \subset A$ as characters given by evaluation at l:

$$\varphi_{\infty,l}(f) = f(l)\,, \quad \forall f \in C_c(X).$$

These characters are all distinct and we thus get a bijection of the space

$$\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*$$

of invertible $\mathbb{Q}$-lattices with the space $\mathcal{E}_\infty$ of extremal KMS_∞ states.

We shall now describe the natural symmetry group of the above system, from an action of the quotient group S

$$S := \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f)$$

as symmetries of our dynamical system. Here the finite adèlic group of GL_2 is given by

$$\mathrm{GL}_2(\mathbb{A}_f) = \prod_{\mathrm{res}} \mathrm{GL}_2(\mathbb{Q}_p),$$

where in the restricted product the p-component lies in $\mathrm{GL}_2(\mathbb{Z}_p)$ for all but finitely many p's. It satisfies

$$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\, \mathrm{GL}_2(R).$$

The action of the subgroup

$$\mathrm{GL}_2(R) \subset S$$

is defined in a straightforward manner using the following right action of $\mathrm{GL}_2(R)$ on $\mathbb{Q}$-lattices:

$$(\Lambda, \phi).\gamma = (\Lambda, \phi \circ \gamma), \quad \forall \gamma \in \mathrm{GL}_2(R).$$

By construction this action preserves the commensurability relation for pairs of $\mathbb{Q}$-lattices and preserves the value of the ratio of covolumes for such pairs. We can view it as the action

$$(\rho, \tau).\gamma = (\rho \circ \gamma, \tau)$$

of $\mathrm{GL}_2(R)$ on $Y = M_2(R) \times \mathbb{H}$, which commutes with the left action of $\mathrm{GL}_2^+(\mathbb{Q})$. Thus, this action defines automorphisms of the dynamical system (A, σ_t) by

$$\theta_\gamma(f)(g, y) := f(g, y.\gamma), \quad \forall f \in \mathcal{A}, \gamma \in \mathrm{GL}_2(R),$$

and one has

$$\theta_{\gamma_1}\, \theta_{\gamma_2} = \theta_{\gamma_1 \gamma_2}, \quad \forall \gamma_j \in \mathrm{GL}_2(R).$$

The complementary action of $\mathrm{GL}_2^+(\mathbb{Q})$ is more subtle and is given by endomorphisms of the dynamical system (A, σ_t), following Definition 1.3.

For $m \in M_2(\mathbb{Z})^+$, let $\tilde{m} = \mathrm{Det}(m)\, m^{-1} \in M_2(\mathbb{Z})^+$. The range R_m of the map $\rho \to \rho\, \tilde{m}$ only depends on $L = m(\mathbb{Z}^2)$. Indeed if $m_j \in M_2(\mathbb{Z})^+$ fulfill $m_1(\mathbb{Z}^2) = m_2(\mathbb{Z}^2)$ then $m_2 = m_1 \gamma$ for some $\gamma \in \Gamma$, hence $M_2(R)\, \tilde{m}_1 = M_2(R)\, \tilde{m}_2$. Let then

$$e_L \in C(X) \tag{1.110}$$

be the characteristic function of $\Gamma \backslash (R_m \times \mathbb{H}) \subset \Gamma \backslash (M_2(R) \times \mathbb{H})$, for any m such that $m(\mathbb{Z}^2) = L$. Equivalently, it is the characteristic function of the open and closed subset $E_L \subset X$ of $\mathbb{Q}$-lattices of the form $(\Lambda, \phi \circ \tilde{m})$. One has

$$e_L\, e_{L'} = e_{L \cap L'}, \qquad e_{\mathbb{Z}^2} = 1. \tag{1.111}$$

For $l = (\Lambda, \phi) \in E_L \subset X$ and $m \in M_2(\mathbb{Z})^+$, $m(\mathbb{Z}^2) = L$ we let

$$l \circ \tilde{m}^{-1} := (\Lambda, \phi \circ \tilde{m}^{-1}) \in X.$$

This map preserves commensurability of $\mathbb{Q}$-lattices. On Y it is given by

$$(\rho, \tau) \circ \tilde{m}^{-1} := (\rho \circ \tilde{m}^{-1}, \tau), \quad \forall (\rho, \tau) \in R_m \times \mathbb{H}$$

and it commutes with the left action of $\mathrm{GL}_2^+(\mathbb{Q})$. The formula

$$\theta_m(f)(g, y) := f(g, y \circ \tilde{m}^{-1}), \quad \forall y \in R_m \times \mathbb{H}, \tag{1.112}$$

extended by $\theta_m(f)(g, y) = 0$ for $y \notin R_m \times \mathbb{H}$, defines an endomorphism θ_m of A that commutes with the time evolution σ_t. Notice that $\theta_m(1) = e_L \in M(A)$ is a multiplier of A and that θ_m lands in the reduced algebra A_{e_L}, so that (1.112) is unambiguous. Thus one obtains an action of the semigroup $M_2(\mathbb{Z})^+$ by endomorphisms of the dynamical system (A, σ_t), fulfilling Definition 1.3.

Proposition 1.34 *The above actions of the group* $\mathrm{GL}_2(R) \subset S$ *and of the semigroup* $M_2(\mathbb{Z})^+ \subset S$ *assemble to an action of the group* $S = \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f)$ *as symmetries of the dynamical system* (A, σ_t)*.*

Proof. The construction above applies to give an action by endomorphisms of the semigroup $\mathrm{GL}_2(\mathbb{A}_f) \cap M_2(R)$, which contains both $\mathrm{GL}_2(R)$ and $M_2(\mathbb{Z})^+$. It remains to show that the sub-semigroup $\mathbb{N}^\times \subset M_2(\mathbb{Z})^+$ acts by inner endomorphisms of (A, σ_t). Indeed for any $n \in \mathbb{N}^*$, the endomorphism $\theta_{[n]}$ (where $[n] = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \in M_2(\mathbb{Z})^+$) is inner and implemented by the multiplier $\mu_{[n]} \in M(A)$ which was defined in (1.99) above *i.e.* one has

$$\theta_n(f) = \mu_{[n]}\, f\, \mu_{[n]}^*\,, \quad \forall f \in A\,.$$

□

1.9 The subalgebra $\mathcal{A}_\mathbb{Q}$ and the Modular Field

The strategy outlined in §1.6 allows us to find, using Eisenstein series, a suitable *arithmetic* subalgebra $\mathcal{A}_\mathbb{Q}$ of the algebra of unbounded multipliers of the basic Hecke C^*-algebra A of the previous section. The extremal KMS_∞ states $\varphi \in \mathcal{E}_\infty$ extend to $\mathcal{A}_\mathbb{Q}$ and the image $\varphi(\mathcal{A}_\mathbb{Q})$ generates, in the generic case, a specialization $F_\varphi \subset \mathbb{C}$ of the modular field F. The state φ will then intertwine the symmetry group S of the system (A, σ_t) with the Galois group of the modular field *i.e.* we shall show that there exists an isomorphism θ of S with $\mathrm{Gal}(F_\varphi/\mathbb{Q})$ such that

$$\alpha \circ \varphi = \varphi \circ \theta^{-1}(\alpha)\,, \quad \forall \alpha \in \mathrm{Gal}(F_\varphi/\mathbb{Q})\,. \tag{1.113}$$

Let us first define $\mathcal{A}_\mathbb{Q}$ directly without any reference to Eisenstein series and check directly its algebraic properties. We let $Z \subset \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}) \times_\Gamma Y$ be as above and $f \in C(Z)$ be a function with *finite support* in the variable $g \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q})$. Such an f defines an unbounded multiplier of the C^*-algebra A with the product given as above by

$$(f_1 * f_2)(g, y) := \sum_{h \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}), hy \in Y} f_1(gh^{-1}, hy)\; f_2(h, y).$$

One has $Y = M_2(R) \times \mathbb{H}$ and we write $f(g, y) = f(g, \rho, z)$, with $(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(R) \times \mathbb{H}$. In order to define the *arithmetic* elements $f \in \mathcal{A}_\mathbb{Q}$ we first look at the way f depends on $\rho \in M_2(R)$. Let as above $p_N : M_2(R) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. It is a ring homomorphism. We say that f has level N iff $f(g, \rho, z)$ only depends upon $(g, p_N(\rho), z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{H}$. Then specifying f amounts to assigning the finitely many continuous functions $f_{g,m} \in C(\mathbb{H})$ with $m \in M_2(\mathbb{Z}/N\mathbb{Z})$ and

$$f(g, \rho, z) = f_{g, p_N(\rho)}(z).$$

The invariance condition

$$f(g\,\gamma, y) = f(g, \gamma\, y)\,, \quad \forall \gamma \in \Gamma\,,\ g \in \mathrm{GL}_2^+(\mathbb{Q})\,,\ y \in Y \tag{1.114}$$

then shows that

$$f_{g,m}|\gamma = f_{g,m}\,, \quad \forall \gamma \in \Gamma(N) \cap g^{-1}\Gamma g,$$

with standard notations for congruence subgroups and for the slash operation in weight 0 (*cf.* (1.141)).

We denote by F the field of modular functions which are rational over $\mathbb{Q}^{ab}$, *i.e.* the union of the fields F_N of modular functions of level N rational over $\mathbb{Q}(e^{2\pi i/N})$. Its elements are modular functions $h(\tau)$ whose $q^{\frac{1}{N}}$-expansion has all its coefficients in $\mathbb{Q}(e^{2\pi i/N})$ (*cf.* [51]).

The first requirement for arithmetic elements is that

$$f_{g,m} \in F \quad \forall (g, m). \tag{1.115}$$

This condition alone, however, is not sufficient. In fact, the modular field F_N of level N contains (*cf.* [51]) a primitive N-th root of 1. Thus, the condition (1.115) alone allows the algebra $\mathcal{A}_{\mathbb{Q}}$ to contain the cyclotomic field $\mathbb{Q}^{ab} \subset \mathbb{C}$, but this would prevent the existence of "fabulous states", because the "fabulous" property would not be compatible with $\mathbb{C}$-linearity. We shall then impose an additional condition, which forces the spectrum of the corresponding elements of $\mathcal{A}_{\mathbb{Q}}$ to contain all Galois conjugates of such a root, so that no such element can be a scalar. This is, in effect, a consistency condition on the roots of unity that appear in the coefficients of the q-series, when ρ is multiplied on the left by a diagonal matrix.

Consider elements $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and $\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, respectively of the form

$$g = r \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \alpha = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \tag{1.116}$$

with k prime to N and $n|N$.

Definition 1.35 *We shall say that f of level N_0 is* arithmetic *($f \in \mathcal{A}_{\mathbb{Q}}$) iff for any multiple N of N_0 and any pair (g, α) as in* (1.116) *we have $f_{g,m} \in F_N$ for all $m \in M_2(\mathbb{Z}/N\mathbb{Z})$ and the q-series of $f_{g,\alpha m}$ is obtained from the q-series for $f_{g,m}$ by raising to the power k the roots of unity that appear as coefficients.*

The arithmetic subalgebra $\mathcal{A}_{\mathbb{Q}}$ enriches the structure of the noncommutative space to that of a "noncommutative arithmetic variety". As we shall prove in Theorem 1.39, a generic ground state φ of the system, when evaluated on $\mathcal{A}_{\mathbb{Q}}$ generates an embedded copy F_φ of the modular field in $\mathbb{C}$. Moreover, there exists a unique isomorphism $\theta = \theta_\varphi$ of the symmetry group S of the system with $\mathrm{Gal}(F_\varphi/\mathbb{Q})$, such that

$$\theta(\sigma) \circ \varphi = \varphi \circ \sigma\,, \quad \forall \sigma \in S\,.$$

A first step towards this result is to show that the arithmeticity condition is equivalent to a covariance property under left multiplication of ρ by elements $\alpha \in \mathrm{GL}_2(R)$, in terms of Galois automorphisms. The condition is always satisfied for $\alpha \in \mathrm{SL}_2(R)$.

For each $g \in \mathrm{GL}_2(\mathbb{A}_f)$, we let $\mathrm{Gal}(g) \in \mathrm{Aut}(F)$ be its natural action on F, written in a covariant way so that

$$\mathrm{Gal}(g_1\, g_2) = \mathrm{Gal}(g_1) \circ \mathrm{Gal}(g_2).$$

With the standard contravariant notation $f \mapsto f^g$ (*cf. e.g.* [28]) we let, for all $f \in F$,

$$\mathrm{Gal}(g)(f) := f^{\tilde{g}}, \quad \tilde{g} = \mathrm{Det}(g)\, g^{-1}. \tag{1.117}$$

Lemma 1.36 *For any $\alpha \in \mathrm{SL}_2(R)$ one has*

$$f_{g,\alpha\, m} = \mathrm{Gal}(\alpha) f_{g',m},$$

where $g\,\alpha = \alpha'\, g'$ is the decomposition of $g\,\alpha$ as a product in $\mathrm{GL}_2(R).\mathrm{GL}_2(\mathbb{Q})$.

Proof. Notice that the decomposition $\alpha'\, g'$ is not unique, but the left invariance

$$f(\gamma\, g', \rho, \tau) = f(g', \rho, \tau), \quad \forall \gamma \in \Gamma$$

shows that the above condition is well defined. Let $p_N : M_2(R) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the projection. Then $f_{g,\alpha\, m} = f_{g,p_N(\alpha)p_N(m)}$, for f of level N. Let $\gamma \in \Gamma$ be such that $p_N(\gamma) = p_N(\alpha)$. Then

$$f_{g,\alpha m}(\tau) = f(g, \gamma m, \tau) = f(g\gamma, m, \gamma^{-1}(\tau)).$$

Thus, for $g' = g\gamma$, one obtains the required condition. □

Lemma 1.37 *A function f is in $\mathcal{A}_{\mathbb{Q}}$ iff condition* (1.115) *is satisfied and*

$$f_{g,\alpha\, m} = \mathrm{Gal}(\alpha) f_{g',m}, \quad \forall \alpha \in \mathrm{GL}_2(R), \tag{1.118}$$

where $g\,\alpha = \alpha'\, g'$ is the decomposition of $g\,\alpha$ as a product in $\mathrm{GL}_2(R).\mathrm{GL}_2(\mathbb{Q})$.

Proof. By Lemma 1.36, the only nontrivial part of the covariance condition (1.118) is the case of diagonal matrices $\delta = \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ with $u \in \mathrm{GL}_1(R)$.

To prove (1.118) we can assume that $g = g_0\,\gamma$ with g_0 diagonal as in (1.116) and $\gamma \in \Gamma$. Let then $\gamma\,\alpha = \delta\,\alpha_1$ with δ as above and $\alpha_1 \in \mathrm{SL}_2(R)$. One has

$$f_{g_0,\delta\alpha_1 m} = \mathrm{Gal}(\delta)\, f_{g_0,\alpha_1 m}$$

by Definition 1.35 since $\mathrm{Gal}(\delta)$ is given by raising the roots of unity that appear as coefficients of the q-expansion to the power k where u is the residue of k modulo N (*cf.* [51] (6.2.1) p.141). One then has

$$f_{g,\alpha\, m} = \mathrm{Gal}(\gamma^{-1}) f_{g_0,\gamma\alpha m} = \mathrm{Gal}(\gamma^{-1}\delta)\, f_{g_0,\alpha_1 m}$$

and by Lemma 1.36, with $g_0\alpha_1 = \alpha_1' g_0'$ we get

$$f_{g,\alpha\, m} = \mathrm{Gal}(\gamma^{-1}\delta\alpha_1)\, f_{g_0',m} = \mathrm{Gal}(\alpha)\, f_{g_0',m}$$

Moreover

$$g\,\alpha = g_0\,\gamma\,\alpha = g_0\,\delta\,\alpha_1 = \delta\, g_0\,\alpha_1 = \delta\,\alpha_1'\, g_0'$$

which shows that $g' = g_0'$
One checks similarly that the converse holds. $\square$

Proposition 1.38 *$\mathcal{A}_{\mathbb{Q}}$ is a subalgebra of the algebra of unbounded multipliers of A, globally invariant under the action of the symmetry group S.*

Proof. For each *generic* value of $\tau \in \mathbb{H}$ the evaluation map

$$h \in F \mapsto I_\tau(h) = h(\tau) \in \mathbb{C}$$

gives an isomorphism of F with a subfield $F_\tau \subset \mathbb{C}$ and a corresponding action Gal_τ of $\mathrm{GL}_2(\mathbb{A}_f)$ by automorphisms of F_τ, such that

$$\mathrm{Gal}_\tau(g)(I_\tau(h)) = I_\tau(\mathrm{Gal}(g)(h)). \tag{1.119}$$

We first rewrite the product as

$$(f_1 * f_2)(g,\rho,\tau) = \sum_{g_1 \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, g_1\rho \in M_2(R)} f_1(gg_1^{-1}, g_1\rho, g_1(\tau))\, f_2(g_1,\rho,\tau)\,.$$

The proof that $(f_1 * f_2)_{g,m} \in F$ is the same as in Proposition 2 of ([10]). It remains to be shown that condition (1.118) is stable under convolution. Thus we let $\alpha \in \mathrm{GL}_2(R)$ and we want to show that $f_1 * f_2$ fulfills (1.118). We let $g' \in \mathrm{GL}_2(\mathbb{Q})$ and $\beta \in \mathrm{GL}_2(R)$ with $g\,\alpha = \beta\, g'$. By definition, one has

$$(f_1 * f_2)(g,\alpha\,\rho,\tau) = \sum_{g_1 \in \Gamma\backslash \mathrm{GL}_2^+(\mathbb{Q}),\, g_1\alpha\rho \in M_2(R)} f_1(gg_1^{-1}, g_1\,\alpha\,\rho, g_1(\tau))\, f_2(g_1,\alpha\,\rho,\tau).$$

We let $g_1\alpha = \alpha' g_1'$ be the decomposition of $g_1\alpha$, and use 1.118 to write the r.h.s. as

$$(f_1 * f_2)(g,\alpha\,\rho,\tau) = \sum f_1(gg_1^{-1}, g_1\,\alpha\,\rho, g_1(\tau))\, \mathrm{Gal}_\tau(\alpha)(f_2(g_1',\,\rho,\tau))\,,$$

with Gal_τ as in (1.119). The result then follows from the equality

$$f_1(gg_1^{-1}, g_1\,\alpha\,\rho, g_1(\tau)) = \mathrm{Gal}_\tau(\alpha)(f_1(g^{'} g_1^{'\,-1}, g_1'\,\rho, g_1'(\tau))), \tag{1.120}$$

which we now prove. The equality $g_1\alpha = \alpha' g_1'$ together with (1.118) shows that

$$f_1(gg_1^{-1}, g_1\,\alpha\,\rho, g_1(\tau)) = \mathrm{Gal}_{g_1(\tau)}(\alpha')(f_1(g^{'} g_1^{'\,-1}, g_1'\,\rho, g_1(\tau)))\,,$$

using $\quad gg_1^{-1}\,\alpha' = gg_1^{-1}(g_1\alpha\, g_1^{'\,-1}) = g\alpha\, g_1^{'\,-1} = \beta\, g'\; g_1^{'\,-1}.$

For any $h \in F$, one has

$$I_{g_1(\tau)}(\mathrm{Gal}(\alpha')h) = I_{g_1(\tau)}(\mathrm{Gal}(g_1)\mathrm{Gal}(\alpha)\mathrm{Gal}(g_1^{'-1})(h))$$

and, by construction of the Galois action [51],

$$I_{g_1(\tau)} \circ \mathrm{Gal}(g_1) = I_\tau,$$

so that in fact

$$\mathrm{Gal}_{g_1(\tau)}(\alpha')I_{g_1(\tau)}(h) = \mathrm{Gal}_\tau(\alpha)I_{g_1'(\tau)}(h))\,.$$

This proves (1.120) and it shows that $\mathcal{A}_\mathbb{Q}$ is a subalgebra of the algebra of unbounded multipliers of A. To prove the invariance under S is straightforward, since the endomorphisms are all acting on the ρ variable by right multiplication, which does not interfere with condition (1.118). $\Box$

In fact, modulo the nuance between "forms" and functions, the above algebra $\mathcal{A}_\mathbb{Q}$ is intimately related to the modular Hecke algebra of [10].

We can now state the main result extending Theorem 1.7 to the two dimensional case.

Theorem 1.39 *Let $l = (\rho, \tau)$ be a generic invertible $\mathbb{Q}$-lattice and $\varphi_l \in \mathcal{E}_\infty$ be the corresponding KMS$_\infty$ state. The image $\varphi_l(\mathcal{A}_\mathbb{Q}) \subset \mathbb{C}$ generates the specialization $F_\tau \subset \mathbb{C}$ of the modular field F obtained for the modulus τ. The action of the symmetry group S of the dynamical system (A, σ_t) is intertwined by φ with the Galois group of the modular field F_τ by the formula*

$$\varphi \circ \alpha = \mathrm{Gal}_\tau(\rho\,\alpha\,\rho^{-1}) \circ \varphi.$$

Proof. We first need to exhibit enough elements of $\mathcal{A}_\mathbb{Q}$. Let us first deal with functions $f(g, \rho, \tau)$ which vanish except when $g \in \Gamma$. By construction these are functions on the space X of $\mathbb{Q}$-lattices

$$X = (\text{Space of } \mathbb{Q}\text{-lattices in } \mathbb{C})/\mathbb{C}^* \sim \Gamma\backslash(M_2(R) \times \mathbb{H})\,. \tag{1.121}$$

To obtain such elements of $\mathcal{A}_\mathbb{Q}$ we start with Eisenstein series and view them as functions on the space of $\mathbb{Q}$-lattices. Recall that to a pair $(\rho, \tau) \in Y$ we associate the $\mathbb{Q}$-lattice $(\Lambda, \phi) = \theta(\rho, \tau)$ by

$$\Lambda = \mathbb{Z} + \tau\,\mathbb{Z}\,,\ \phi(a) = \rho_1(a) - \tau\rho_2(a) \in \mathbb{Q}\Lambda/\Lambda, \tag{1.122}$$

where $\rho_j(a) = \sum \rho_{jk}(a_k) \in \mathbb{Q}/\mathbb{Z}$, for $a = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2$. The Eisenstein series are given by

$$E_{2k,a}(\rho, \tau) = \pi^{-2k} \sum_{y \in \Lambda + \phi(a)} y^{-2k}. \tag{1.123}$$

This is undefined when $\phi(a) \in \Lambda$, but we shall easily deal with that point below. For $k = 1$ we let

$$X_a(\rho, \tau) = \pi^{-2} \left(\sum_{y \in \Lambda + \phi(a)} y^{-2} - {\sum_{y \in \Lambda}}' y^{-2} \right) \tag{1.124}$$

when $\phi(a) \notin \Lambda$ and $X_a(\rho, \tau) = 0$ if $\phi(a) \in \Lambda$. This is just the evaluation of the Weierstrass $\wp$-function on $\phi(a)$.

For $\gamma \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$ we have $\quad X_a(\gamma\,\rho, \gamma\,\tau) = (c\tau + d)^2 \, X_a(\rho, \tau), \quad$ which shows that the function $c(\tau) X_a$ is Γ-invariant on Y, where

$$c(\tau) = -2^7 \, 3^5 \, \frac{g_2 g_3}{\Delta} \tag{1.125}$$

has weight -2 and no pole in $\mathbb{H}$. We use c as we used the covolume in the 1-dimensional case, to pass to modular functions. This corresponds in weight 2 to passing from division values of the Weierstrass $\wp$-function to the Fricke functions (*cf.* [28] §6.2)

$$f_v(\tau) = -2^7 3^5 \, \frac{g_2 \, g_3}{\Delta} \wp(\lambda(v, \tau)), \tag{1.126}$$

where $v = (v_1, v_2) \in (\mathbb{Q}/\mathbb{Z})^2$ and $\lambda(v, \tau) := v_1 \, \tau + v_2$. Here $g_2, \ g_3$ are the coefficients giving the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ in Weierstrass form,

$$y^2 = 4x^3 - g_2 x - g_3,$$

with discriminant $\Delta = g_2^3 - 27 g_3^2$. One has (up to powers of π)

$$g_2 = 60 \, e_4 \, , \quad g_3 = 140 \, e_6,$$

where one defines the standard modular forms of even weight $k \in 2\,\mathbb{N}$ as

$$e_k(\Lambda) := \pi^{-k} \sum_{y \in \Lambda \setminus \{0\}} y^{-k}$$

with q-expansion ($q = e^{2\pi i \tau}$)

$$e_k = \frac{2^k}{k!} B_{\frac{k}{2}} + (-1)^{k/2} \frac{2^{k+1}}{(k-1)!} \sum_{1}^{\infty} \sigma_{k-1}(N) \, q^N,$$

where the B_n are the Bernoulli numbers and $\sigma_n(N)$ is the sum of d^n over the divisors d of N. The e_{2n} for $n \geq 2$ are in the ring $\mathbb{Q}[e_4, e_6]$ (cf [56]) thanks to the relation

$$\frac{1}{3}(m-3)(4m^2 - 1) e_{2m} = \sum_{2}^{m-2} (2r-1)(2m-2r-1) e_{2r} e_{2m-2r}. \tag{1.127}$$

Notice that $\mathcal{X}_a := c\,X_a \in C(M_2(R)\times\mathbb{H}) = C(Y)$ is a continuous function on Y. The continuity of X_a as a function of ρ comes from the fact that it only involves the restriction $\rho_N \in M_2(\mathbb{Z}/N\mathbb{Z})$ of ρ to N-torsion elements a with $Na = 0$.
We view $\mathcal{X}_a$ as a function on Z by

$$\mathcal{X}_a(\gamma,\rho,\tau) := \mathcal{X}_a(\rho,\tau)\,, \quad \forall \gamma\in\Gamma,$$

while it vanishes for $\gamma \notin \Gamma$. Let us show that $\mathcal{X}_a \in \mathcal{A}_{\mathbb{Q}}$. Since the Fricke functions belong to the modular field F we only need to check (1.118). For $\alpha\in \mathrm{GL}_2(R)$ and generic τ we want to show that

$$\mathcal{X}_a(\alpha\,\rho,\tau) = \ \mathrm{Gal}_\tau(\alpha)\mathcal{X}_a(\rho,\tau).$$

If $\rho(a)=0$ both sides vanish, otherwise they are both given by Fricke functions f_v, $f_{v'}$, corresponding respectively to the labels (using (1.122))

$$v = s\,\alpha\,\rho(a)\,, \quad v' = s\,\rho(a)\,, \quad s = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Thus, $v' = s\,\alpha\,s^{-1}(v) = (s^{-1}\,\alpha^t\,s)^t(v)$ and the result follows from (1.117) and the equality

$$\tilde{\alpha} = \ \mathrm{Det}(\alpha)\ \alpha^{-1} = \ s^{-1}\,\alpha^t\,s,$$

with the Galois group $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\pm 1$ of the modular field F_n over $\mathbb{Q}(j)$ acting on the Fricke functions by permutation of their labels:

$$f_v^{\sigma(u)} = f_{u^t v}\,, \quad \forall u\in \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}).$$

This shows that $\mathcal{X}_a \in \mathcal{A}_{\mathbb{Q}}$ and it suffices to show that, with the notation of the theorem, $\varphi_l(\mathcal{A}_{\mathbb{Q}})$ generates F_τ, since the modular field F is the field generated over $\mathbb{Q}$ by all the Fricke functions. It already contains $\mathbb{Q}(j)$ at level 2 and it contains in fact $\mathbb{Q}^{ab}(j)$.

Let us now display elements $T_{r_1,r_2} \in \mathcal{A}_{\mathbb{Q}}$, $r_j \in \mathbb{Q}^*_+, r_1|r_2$, associated to the classical Hecke correspondences. We let $C_{r_1,r_2} \subset \Gamma\backslash \mathrm{GL}_2(\mathbb{Q})^+$ be the finite subset given by the double class of $\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ in $\Gamma\backslash\mathrm{GL}_2(\mathbb{Q})^+/\Gamma$. We then define

$$T_{r_1,r_2}(g,\rho,\tau) = 1 \quad \text{if} \quad g\in C_{r_1,r_2}\,,\ g\rho\in M_2(R)\,, \quad T_{r_1,r_2}(g,\rho,\tau)=0 \quad \text{otherwise}\,.$$

One needs to check (1.118), but if $g\,\alpha = \alpha'\,g'$ is the decomposition of $g\,\alpha$ as a product in $\mathrm{GL}_2(R).\mathrm{GL}_2(\mathbb{Q})^+$, then g' belongs to the double coset of $g\in\Gamma\backslash\mathrm{GL}_2(\mathbb{Q})^+/\Gamma$, which gives the required invariance. It is not true that the $T_{r_1,r_2}\in\mathcal{A}_{\mathbb{Q}}$ fulfill the relations of the Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{Q})^+,\Gamma)$ of double cosets, but this holds when r_1, r_2 are restricted to vary among positive *integers*. To see this one checks that the map

$$\tau(f)(g,y) := \ f(g) \quad \text{if} \quad g\in M_2(\mathbb{Z})^+\,, \quad \tau(f)(g,y) = 0 \quad \text{otherwise}$$

defines an isomorphism

$$\tau : \mathcal{H}(M_2(\mathbb{Z})^+, \Gamma) \to \mathcal{A}_{\mathbb{Q}} \tag{1.128}$$

of the standard Hecke algebra $\mathcal{H}(M_2(\mathbb{Z})^+, \Gamma)$ of Γ-biinvariant functions (with Γ-finite support) on $M_2(\mathbb{Z})^+$ with a subalgebra $\mathcal{H} \subset \mathcal{A}_{\mathbb{Q}}$. Notice that it is only because the condition $h\,y \in Y$ of definition (1.89) is now automatically satisfied that τ is a homomorphism.

Let us now show the intertwining equality

$$\varphi_l \circ \alpha = \operatorname{Gal}_\tau(\rho\,\alpha\,\rho^{-1}) \circ \varphi_l\,, \quad \forall \alpha \in S\,. \tag{1.129}$$

One has

$$\varphi_l(f) = f(1, \rho, \tau)\,, \quad \forall f \in \mathcal{A}_{\mathbb{Q}}$$

It is enough to prove (1.129) for $\alpha \in \mathrm{GL}_2(R)$ and for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$.

For $\alpha \in \mathrm{GL}_2(R)$, the state $\varphi_l \circ \alpha$ is given simply by

$$(\varphi_l \circ \alpha)(f) = f(1, \rho\,\alpha, \tau)\,, \quad \forall f \in \mathcal{A}_{\mathbb{Q}}\,,$$

and using (1.118) one gets (1.129) in that case.

Let $m \in M_2^+(\mathbb{Z})$. Then the state $\varphi_l \circ m$ is more tricky to obtain, since it is not the straight composition but the 0-temperature limit of the states obtained by composition of the KMS_β state $\varphi_{l,\beta}$ with the endomorphism θ_m defined in (1.112). Indeed, the range of θ_m is the reduced algebra by the projection e_L, with $L = m(\mathbb{Z}^2)$, on which any of the zero temperature states vanishes identically.

Let us first show that for finite β we have

$$\varphi_{l,\beta} \circ \theta_m = \varphi_{l,\beta}(e_L)\,\varphi_{l',\beta}, \tag{1.130}$$

where $L = m(\mathbb{Z}^2)$ and l' is given by

$$l' = (\rho',\, m'^{-1}(\tau))\,, \quad \rho\,m = m'\,\rho' \in M_2^+(\mathbb{Z}).\mathrm{GL}_2(R)\,. \tag{1.131}$$

By (1.97) we have

$$\varphi_{\beta,l}(\theta_m(f)) = Z^{-1} \sum_{\Gamma\backslash M_2(\mathbb{Z})^+} f(1, \mu\,\rho\,\tilde{m}^{-1}, \mu(\tau))\,\mathrm{Det}(\mu)^{-\beta}\,,$$

where $\mu \in M_2(\mathbb{Z})^+$ is subject to the condition $\mu\,\rho\,\tilde{m}^{-1} \in M_2(R)$. The other values of μ a priori involved in the summation (1.97) do not contribute, since they correspond to the orthogonal of the support of $\theta_m(f)$.

One has $\mathrm{Det}(m) = \mathrm{Det}(m')$ by construction, hence

$$\rho\,\tilde{m}^{-1} = \rho\,m\,\mathrm{Det}(m)^{-1} = \mathrm{Det}(m')^{-1}\,m'\,\rho' = \tilde{m'}^{-1}\,\rho'\,.$$

Therefore the condition $\mu\,\rho\,\tilde{m}^{-1} \in M_2(R)$ holds iff $\mu = \nu\,\tilde{m'}$ for some $\nu \in M_2(\mathbb{Z})^+$. Thus, since $\mathrm{Det}(\mu) = \mathrm{Det}(\nu)\cdot\mathrm{Det}(\tilde{m}')$, we can rewrite, up to multiplication by a scalar,

$$\varphi_{\beta,l}(\theta_m(f)) = Z^{'\,-1} \sum_{\Gamma\backslash M_2(\mathbb{Z})^+} f(1,\nu\,\rho',\nu\,\tilde{m'}(\tau))\,\mathrm{Det}(\nu)^{-\beta}\,.$$

This proves (1.130). It remains to show that on $\mathcal{A}_{\mathbb{Q}}$ we have

$$\varphi_{l'}(f) = \mathrm{Gal}_\tau(\rho\,m\,\rho^{-1})\circ\varphi_l(f)\,,\quad \forall f\in\mathcal{A}_{\mathbb{Q}}\,.$$

Both sides only involve the values of f on invertible $\mathbb{Q}$-lattices, and there, by (1.118) one has

$$f(1,\,\alpha,\tau) = \mathrm{Gal}_\tau(\alpha)f(1,\,1,\tau)\,,\quad \forall\alpha\in\mathrm{GL}(2,R)\,.$$

Thus, we obtain

$$\varphi_{l'}(f) = f(1,\rho',\,m^{'\,-1}(\tau)) = I_{m^{'\,-1}(\tau)}(\mathrm{Gal}(\rho')f) = I_\tau(\mathrm{Gal}(m'\,\rho')f)\,.$$

Since $m'\,\rho' = \rho\,m$, this gives $\mathrm{Gal}_\tau(\rho\,m\,\rho^{-1})\circ\varphi_l(f)$ as required. $\Box$

We shall now work out the algebraic relations fulfilled by the $\mathcal{X}_a$ as extensions of the division formulas of elliptic functions.

We first work with lattice functions of some weight k, or equivalently with forms $f(g,y)\,dy^{k/2}$, and then multiply them by a suitable factor to make them homogeneous of weight 0 under scaling. The functions of weight 2 are the generators, the higher weight ones will be obtained from them by universal formulas with modular forms as coefficients.

The powers X_a^m of the function X_a are then expressed as universal polynomials with coefficients in the ring $\mathbb{Q}\,[e_4,e_6]$ in the following weight $2k$ functions ($k>1$):

$$E_{2k,a}(\rho,\tau) = \pi^{-2k}\sum_{y\in\Lambda+\phi(a)} y^{-2k}. \tag{1.132}$$

These fulfill by construction ([56]) the relations

$$\begin{aligned} E_{2m,a} \;&=\; X_a(E_{2m-2,a}-e_{2m-2}) + \left(1-\binom{2m}{2}\right)e_{2m} \\ &-\; \sum_1^{m-2}\binom{2k+1}{2k}\,e_{2k+2}(E_{2m-2k-2,a}-e_{2m-2k-2})\,. \end{aligned} \tag{1.133}$$

These relations dictate the value of $E_{2k}(\rho,\tau)$ when $\varphi(a)\in\Lambda$: one gets

$$E_{2k}(\rho,\tau) = \nu_{2k}(\tau) \qquad \text{if } \varphi(a)\in\Lambda, \tag{1.134}$$

where ν_{2k} is a modular form of weight $2k$ obtained by induction from (1.133), with X_a replaced by 0 and E_{2m} by ν_{2m}. One has $\nu_{2k} \in \mathbb{Q}\,[e_4, e_6]$ and the first values are

$$\nu_4 = -5\, e_4\,, \quad \nu_6 = -14\, e_6\,, \quad \nu_8 = \frac{45}{7}\, e_4^2\,, \ldots \tag{1.135}$$

We shall now write the important algebraic relations between the functions X_a, which extend the division relations of elliptic functions from invertible $\mathbb{Q}$-lattices to arbitrary ones.

In order to work out the division formulas for the Eisenstein series $E_{2m,a}$ we need to control the image of $\left(\frac{1}{N}\,\mathbb{Z}\right)^2 = \frac{1}{N}\mathbb{Z}^2$ under an arbitrary element $\rho \in M_2(R)$. This is done as follows using the projections π_L defined in (1.65).

Lemma 1.40 *Let $N \in \mathbb{N}^*$, and $\rho \in M_2(R)$. There exists a smallest lattice $L \subset \mathbb{Z}^2$ with $L \supset N\mathbb{Z}^2$, such that $\pi_L(\rho) = 1$. One has*

$$\rho(\frac{1}{N}\,\mathbb{Z}^2) = \frac{1}{N}\,L\,.$$

Proof. There are finitely many lattices L with $N\mathbb{Z}^2 \subset L \subset \mathbb{Z}^2$. Thus, the intersection of those L for which $\pi_L(\rho) = 1$ is still a lattice and fulfills $\pi_L(\rho) = 1$ by (1.66). Let L be this lattice, and let us show that $\rho(\frac{1}{N}\mathbb{Z}^2) \subset \frac{1}{N}\,L$. Let $m \in M_2(\mathbb{Z})^+$ be such that $m(\mathbb{Z}^2) = L$. Then $\pi_L(\rho) = 1$ implies that $\rho = m\mu$ for some $\mu \in M_2(R)$. Thus $\rho(\frac{1}{N}\mathbb{Z}^2) \subset m(\frac{1}{N}\mathbb{Z}^2) = \frac{1}{N}\,L$. Conversely let $L' \subset \mathbb{Z}^2$ be defined by $\rho(\frac{1}{N}\mathbb{Z}^2) = \frac{1}{N}\,L'$. We need to show that $\pi_{L'}(\rho) = 1$, *i.e.* that there exists $m' \in M_2(\mathbb{Z})^+$ such that $L' = m'(\mathbb{Z}^2)$ and ${m'}^{-1}\,\rho \in M_2(R)$. Replacing ρ by $\gamma_1\rho\,\gamma_2$ for $\gamma_j \in \mathrm{SL}_2(\mathbb{Z})$ does not change the problem, hence we can use this freedom to assume that the restriction of ρ to $\left(\frac{1}{N}\,\mathbb{Z}\right)^2$ is of the form

$$\rho_N = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\,, \quad d_j \mid N\,,\ d_1 \mid d_2\,. \tag{1.136}$$

One then takes $m' = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \in M_2(\mathbb{Z})^+$ and checks that $L' = m'(\mathbb{Z}^2)$ while ${m'}^{-1}\rho$ belongs to $M_2(R)$. $\Box$

Given an integer $N > 1$, we let S_N be the set of lattices

$$N\,\mathbb{Z}^2 \subset L \subset \mathbb{Z}^2, \tag{1.137}$$

which is the same as the set of subgroups of $(\mathbb{Z}/N\mathbb{Z})^2$. For each $L \in S_N$ we define a projection $\pi(N, L)$ by

$$\pi(N, L) = \pi_L \prod_{L' \in S_N, L' \subsetneq L} (1 - \pi_{L'})\,. \tag{1.138}$$

By Lemma 1.40 the range of $\pi(N, L)$ is exactly the set of $\rho \in M_2(R)$ such that

$$\rho(\frac{1}{N}\,\mathbb{Z}^2) = \frac{1}{N}\,L\,. \tag{1.139}$$

The general form of the division relations is as follows.

Proposition 1.41 *There exists canonical modular forms $\omega_{N,L,k}$ of level N and weight $2k$, such that for all k and $(\rho,\tau)\in Y$ they satisfy*

$$\sum_{Na=0} X_a^k(\rho,\tau) = \sum_{L\in S_N} \pi(N,L)(\rho)\,\omega_{N,L,k}(\tau)\,.$$

In fact, we shall give explicit formulas for the $\omega_{N,L,k}$ and show in particular that

$$\omega_{N,L,k}(\gamma\,\tau) = (c\tau+d)^{2k}\,\omega_{N,\gamma^{-1}L,k}(\tau),$$

which implies that $\omega_{N,L,k}$ is of level N.

We prove it for $k=1$ and then proceed by induction on k. The division formulas in weight 2 involve the 1-cocycle on the group $\mathrm{GL}_2^+(\mathbb{Q})$ with values in Eisenstein series of weight 2 given in terms of the Dedekind η-function by (cf. [10])

$$\mu_\gamma\,(\tau) \;=\; \frac{1}{12\pi i}\,\frac{d}{d\tau}\log\frac{\Delta|\gamma}{\Delta} \;=\; \frac{1}{2\pi i}\,\frac{d}{d\tau}\log\frac{\eta^4|\gamma}{\eta^4}\,, \tag{1.140}$$

where we used the standard 'slash operator' notation for the action of $\mathrm{GL}_2^+(\mathbb{R})$ on functions on the upper half plane:

$$f|_k\,\alpha\,(z) = \mathrm{Det}\,(\alpha)^{k/2}\,f(\alpha\cdot z)\,j(\alpha,z)^{-k}\,, \tag{1.141}$$

$$\alpha=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in \mathrm{GL}_2^+(\mathbb{R}),\quad \alpha\cdot z=\frac{az+b}{cz+d}\quad\text{and}\quad j(\alpha,z)\;=\;cz+d\,.$$

Since $\mu_\gamma=0$ for $\gamma\in\Gamma$, the cocycle property

$$\mu_{\gamma_1\cdot\gamma_2}\;=\;\mu_{\gamma_1}|\gamma_2\;+\;\mu_{\gamma_2} \tag{1.142}$$

shows that, for $m\in M_2(\mathbb{Z})^+$, the value of $\mu_{m^{-1}}$ only depends upon the lattice $L=m(\mathbb{Z}^2)$. We shall denote it by μ_L.

Lemma 1.42 *For any integer N, the X_a, $a\in\mathbb{Q}/\mathbb{Z}$, fulfill the relation*

$$\sum_{N\,a=0} X_a \;=\; N^2\sum_{L\in S_N}\pi(N,L)\;\;\mu_L$$

By construction the projections $\pi(N,L)$, $L\in S_N$ form a partition of unity,

$$\sum_{L\in S_N}\pi(N,L) = \;1\,.$$

Thus to prove the lemma it is enough to evaluate both sides on $\rho\in\pi(N,L)$.

We can moreover use the equality

$$\mu_{\gamma^{-1}L}=\mu_L|\gamma\,,\quad \forall\gamma\in\Gamma$$

to assume that L and ρ_N are of the form

$$L = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \mathbb{Z}^2 , \qquad \rho_N = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} , \quad d_j \mid N \, , \; d_1 \mid d_2 \, .$$

Let $d_2 = n\, d_1$. The order of the kernel of ρ_N is $d_1\, d_2$ and the computation of $\sum_{N\, a=0} X_a(\rho, \tau)$ gives

$$-N^2 e_2(\tau) \, + \, d_1\, d_2\, N^2 \sum_{(a,b)\in \mathbb{Z}^2\setminus\{0\}} (a\, d_1 - b\, d_2 \tau)^{-2}$$

which gives $N^2(n\, e_2(n\,\tau) - e_2(\tau)) = N^2 \mu_L$.

This proves the proposition for $k = 1$. Let us proceed by induction using (1.133) to express X_a^k as $E_{2k,a}$ plus a polynomial of degree $< k$ in X_a with coefficients in $\mathbb{Q}[e_4, e_6]$. Thus, we only need to prove the equality

$$\sum_{N\, a=0} E_{2k,a} = \sum_{L\in S_N} \pi(N, L) \;\alpha_{N,L,k},$$

where the modular forms $\alpha_{N,L,k}$ are given explicitly as

$$\alpha_{N,L,k} = \; N^{2k} d^2 n^{k+1} e_{2k} | m^{-1} \; - \mathrm{Det}(m) \; (e_{2k} - \nu_{2k}),$$

with $m \in M_2(\mathbb{Z})^+$, $m(\mathbb{Z}^2) = L$, and (d, dn) the elementary divisor of L.
The proof is obtained as above by evaluating both sides on arbitrary $\rho \in \pi(N, L)$. $\Box$

One can rewrite all the above relations in terms of the weight 0 elements

$$\mathcal{X}_a := \; c\, X_a \, , \quad \mathcal{E}_{2k,a} := \; c^k \, E_{2k,a} \in \mathcal{A}_{\mathbb{Q}}.$$

In particular, the two basic modular functions $c^2\, e_4$ and $c^3\, e_6$ are replaced by

$$c^2\, e_4 = \; \frac{1}{5}\, j\, (j - 1728) \, , \quad c^3\, e_6 = \; -\frac{2}{35}\, j\, (j - 1728)^2.$$

We can now rewrite the relations (1.133) in terms of universal polynomials

$$P_n \in \mathbb{Q}(j)[X],$$

which express the generators $\mathcal{E}_{2k,a}$ in terms of $\mathcal{X}_a$ by

$$\mathcal{E}_{2k,a} = P_k(\mathcal{X}_a).$$

In fact, from (1.133) we see that the coefficients of P_k are themselves polynomials in j rather than rational fractions, so that

$$P_n \in \mathbb{Q}[j, X].$$

The first ones are given by

$$P_2 = X^2 - j(j - 1728) \, , \quad P_3 = X^3 - \frac{9}{5}\, X j(j - 1728) + \frac{4}{5} j(j - 1728)^2 \, , \quad \cdots$$

1.10 The noncommutative boundary of modular curves

We shall explain in this section how to combine the dual of the GL_2-system described above with the idea, originally developed in the work of Connes–Douglas–Schwarz [12] and Manin–Marcolli [37], of enlarging the boundary of modular curves with a noncommutative space that accounts for the degeneration of elliptic curves to noncommutative tori.

The GL_2-system described in the previous sections admits a "dual" system obtained by considering $\mathbb{Q}$-lattices up to commensurability but no longer up to scaling. Equivalently this corresponds to taking the cross product of the GL_2-system by the action of the Pontrjagin dual of $\mathbb{C}^*$, which combines the time evolution σ_t with an action by the group $\mathbb{Z}$ of integral weights of modular forms. The resulting space is the total space of the natural $\mathbb{C}^*$-bundle.

In adélic terms this "dual" noncommutative space $\mathcal{L}_2$ is described as follows,

Proposition 1.43 *There is a canonical bijection from the space of* $\mathrm{GL}_2(\mathbb{Q})$*-orbits of the left action of* $\mathrm{GL}_2(\mathbb{Q})$ *on* $M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})$ *to the space* $\mathcal{L}_2$ *of commensurability classes of two-dimensional* $\mathbb{Q}$*-lattices.*

Proof. The space of $\mathrm{GL}_2(\mathbb{Q})$ orbits on $M_2(\mathbb{A}_f) \times \mathrm{GL}_2(\mathbb{R})$ is the same as the space of $\mathrm{GL}_2^+(\mathbb{Q})$ orbits on $M_2(R) \times \mathrm{GL}_2^+(\mathbb{R})$. □

By the results of the previous sections, the classical space obtained by considering the zero temperature limit of the quantum statistical mechanical system describing commensurability classes of 2-dimensional $\mathbb{Q}$-lattices up to scaling is the Shimura variety that represents the projective limit of all the modular curves

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^* . \tag{1.143}$$

Usually, the Shimura variety is constructed as the projective limit of the

$$\Gamma'\backslash\mathrm{GL}_2(\mathbb{R})^+/\mathbb{C}^*$$

over congruence subgroups $\Gamma' \subset \Gamma$. This gives a connected component in (1.143). The other components play a crucial role in the present context, in that the existence of several connected components allows for non-constant solutions of the equation $\zeta^n = 1$. Moreover all the components are permuted by the Galois covariance property of the arithmetic elements of the GL_2 system.

The total space of the natural $\mathbb{C}^*$-bundle, *i.e.* the quotient

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}), \tag{1.144}$$

is the space of *invertible* 2-dimensional $\mathbb{Q}$-lattices (not up to scaling).

In the GL_1 case, the analog of (1.144), *i.e.* the space of idèle classes

$$\mathrm{GL}_1(\mathbb{Q})\backslash\mathrm{GL}_1(\mathbb{A}),$$

is compactified by first considering the noncommutative space of commensurability classes of $\mathbb{Q}$-lattices not up to scaling

$$\mathcal{L} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}^{\cdot},$$

where $\mathbb{A}^{\cdot}$ is the space of adèles with nonzero archimedean component. The next step, which is crucial in obtaining the geometric space underlying the spectral realization of the zeros of the Riemann zeta function, is to add an additional "stratum" that gives the noncommutative space of adèle classes

$$\overline{\mathcal{L}} = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}\,,$$

which will be analyzed in the next Chapter.

Similarly, in the GL_2 case, the classical space given by the Shimura variety (1.143) is first "compactified" by adding noncommutative "boundary strata" obtained by replacing $\mathrm{GL}_2(\mathbb{A}_f)$ in $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_f)\times\mathrm{GL}_2(\mathbb{R})$ by all matrices $M_2(\mathbb{A}_f)$. As boundary stratum of the Shimura variety it corresponds to degenerating the invertible $\mathbb{Q}$-structure ϕ on the lattice to a non-invertible one and yields the notion of $\mathbb{Q}$-lattice. The corresponding space of commensurability classes of $\mathbb{Q}$-lattices up to scaling played a central role in this whole chapter.

The space of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices (not up to scaling) is

$$\mathcal{L}_2 = \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f)\times\mathrm{GL}_2(\mathbb{R}))\,. \tag{1.145}$$

On $\mathcal{L}_2$ we can consider not just modular functions but all modular forms as functions. One obtains in this way an antihomomorphism of the modular Hecke algebra of level one of [10] (with variable $\alpha\in\mathrm{GL}_2(\mathbb{Q})^+$ restricted to $M_2(\mathbb{Z})^+$) to the algebra of coordinates on $\mathcal{L}_2$.

The further compactification at the archimedean place, corresponding to $\mathcal{L}\hookrightarrow\overline{\mathcal{L}}$ in the GL_1 case, now consists of replacing $\mathrm{GL}_2(\mathbb{R})$ by matrices $M_2(\mathbb{R})$. This corresponds to degenerating the lattices to pseudo-lattices (in the sense of [34]) or in more geometric terms, to a degeneration of elliptic curves to noncommutative tori. It is this part of the "noncommutative compactification" that was considered in [12] and [37].

A $\mathbb{Q}$-pseudolattice in $\mathbb{C}$ is a pair (Λ,ϕ), with $\Lambda = j(\mathbb{Z}^2)$ the image of a homomorphism $j:\mathbb{Z}^2\to\ell$, with $\ell\subset\mathbb{R}^2\cong\mathbb{C}$ a real 1-dimensional subspace, and with a group homomorphism

$$\phi:\mathbb{Q}^2/\mathbb{Z}^2\to\mathbb{Q}\Lambda/\Lambda.$$

The $\mathbb{Q}$-pseudolattice is nondegenerate if j is injective and is invertible if ϕ is invertible.

Proposition 1.44 *Let $\partial Y := M_2(R)\times\mathbb{P}^1(\mathbb{R})$. The map*

$$(\rho,\theta)\mapsto(\Lambda,\phi),\qquad \Lambda=\mathbb{Z}+\theta\mathbb{Z},\quad \phi(x)=\rho_1(x)-\theta\rho_2(x) \tag{1.146}$$

gives an identification

$$\Gamma\backslash\partial Y\simeq(\textit{Space of }\mathbb{Q}\textit{-pseudolattices in }\mathbb{C})/\mathbb{C}^*. \tag{1.147}$$

This space parameterizes the degenerations of 2-dimensional $\mathbb{Q}$*-lattices*

$$\lambda(y) = (\Lambda, \phi) \quad \text{where} \quad \Lambda = \tilde{h}(\mathbb{Z} + i\,\mathbb{Z}) \quad \text{and} \quad \phi = \tilde{h} \circ \rho, \tag{1.148}$$

for $y = (\rho, h) \in M_2(R) \times \mathrm{GL}_2(\mathbb{R})$ *and* $\tilde{h} = h^{-1}\mathrm{Det}\,(h)$*, when* $h \in \mathrm{GL}_2(\mathbb{R})$ *degenerates to a non-invertible matrix in* $M_2(\mathbb{R})$*.*

Proof. $\mathbb{Q}$-pseudolattices in $\mathbb{C}$ are of the form

$$\Lambda = \lambda(\mathbb{Z} + \theta\mathbb{Z}), \quad \phi(a) = \lambda\rho_1(a) - \lambda\theta\rho_2(a), \tag{1.149}$$

for $\lambda \in \mathbb{C}^*$ and $\theta \in \mathbb{P}^1(\mathbb{R})$ and $\rho \in M_2(R)$. The action of $\mathbb{C}^*$ multiplies λ, while leaving θ unchanged. This corresponds to changing the 1-dimensional linear subspace of $\mathbb{C}$ containing the pseudo-lattice and rescaling it. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ by fractional linear transformations changes θ. The non-degenerate pseudolattices correspond to the values $\theta \in \mathbb{P}^1(\mathbb{R}) \smallsetminus \mathbb{P}^1(\mathbb{Q})$ and the degenerate pseudolattices to the cusps $\mathbb{P}^1(\mathbb{Q})$.

For $y = (\rho, h) \in M_2(R) \times \mathrm{GL}_2(\mathbb{R})$ consider the $\mathbb{Q}$-lattice (1.148), for $\tilde{h} = h^{-1}\mathrm{Det}\,(h)$. Here we use the basis $\{e_1 = 1, e_2 = -i\}$ of the $\mathbb{R}$-vector space $\mathbb{C}$ to let $\mathrm{GL}_2^+(\mathbb{R})$ act on $\mathbb{C}$ as $\mathbb{R}$-linear transformations. These formulas continue to make sense when $h \in M_2(\mathbb{R})$ and the image $\Lambda = \tilde{h}(\mathbb{Z} + i\,\mathbb{Z})$ is a pseudolattice when the matrix h is no longer invertible.

To see this more explicitly, consider the right action

$$m \mapsto m \cdot z \tag{1.150}$$

of $\mathbb{C}^*$ on $M_2(\mathbb{R})$ determined by the inclusion $\mathbb{C}^* \subset \mathrm{GL}_2(\mathbb{R})$ as in (1.80), The action of $\mathbb{C}^*$ on $M_2(\mathbb{R}) \smallsetminus \{0\}$ is free and proper. The map

$$\rho(\alpha) = \begin{cases} \alpha(i) & (c,d) \neq (0,0) \\ \infty & (c,d) = (0,0) \end{cases} \quad \text{with} \quad \alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{1.151}$$

defines an isomorphism

$$\rho : (M_2(\mathbb{R}) \smallsetminus \{0\})/\mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}), \tag{1.152}$$

equivariant with respect to the left action of $\mathrm{GL}_2(\mathbb{R})$ on $M_2(\mathbb{R})$ and the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{P}^1(\mathbb{C})$ by fractional linear transformations. Moreover, this maps $M_2(\mathbb{R})^+$ to the closure of the upper half plane

$$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{R}). \tag{1.153}$$

The rank one matrices in $M_2(\mathbb{R})$ map to $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. In fact, the isotropy group of $m \in M_2(\mathbb{R})$ is trivial if $m \neq 0$, since $m \cdot z = m$ only has nontrivial solutions for $m = 0$, since $z - 1$ is invertible when nonzero. This shows that $M_2(\mathbb{R})\backslash\{0\}$ is the total space of a principal $\mathbb{C}^*$-bundle. □

Notice that, unlike the case of $\mathbb{Q}$-lattices of (1.87), where the quotient $\Gamma\backslash Y$ can be considered as a classical quotient, here the space $\Gamma\backslash\partial Y$ should be regarded as a noncommutative space with function algebra

$$C(\partial Y) \rtimes \Gamma.$$

The usual algebro–geometric compactification of a modular curve $Y_{\Gamma'} = \Gamma'\backslash\mathbb{H}$, for Γ' a finite index subgroup of Γ, is obtained by adding the cusp points $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$,

$$X_{\Gamma'} = Y_{\Gamma'} \cup \{\text{ cusps }\} = \Gamma'\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))\,. \tag{1.154}$$

Replacing $\mathrm{GL}_2(\mathbb{R})$ by $M_2(\mathbb{R})$ in (1.145) corresponds to replacing the cusp points $\mathbb{P}^1(\mathbb{Q})$ by the full boundary $\mathbb{P}^1(\mathbb{R})$ of $\mathbb{H}$. Since Γ does not act discretely on $\mathbb{P}^1(\mathbb{R})$, the quotient is best described by noncommutative geometry, as the cross product C^*-algebra $C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma'$ or, up to Morita equivalence,

$$C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma, \tag{1.155}$$

with $\mathbb{P}$ the coset space $\mathbb{P} = \Gamma/\Gamma'$.

The noncommutative boundary of modular curves defined this way retains a lot of the arithmetic information of the classical modular curves. Various results of [37] show, from the number theoretic point of view, why the irrational points of $\mathbb{P}^1(\mathbb{R})$ in the boundary of $\mathbb{H}$ should be considered as part of the compactification of modular curves.

The first such result is that the classical definition of modular symbols (*cf.* [36]), as homology classes on modular curves defined by geodesics connecting cusp points, can be generalized to "limiting modular symbols", which are asymptotic cycles determined by geodesics ending at irrational points. The properties of limiting modular symbols are determined by the spectral theory of the Ruelle transfer operator of a dynamical system, which generalizes the Gauss shift of the continued fraction expansion by taking into account the extra datum of the coset space $\mathbb{P}$.

Manin's modular complex (*cf.* [36]) gives a combinatorial presentation of the first homology of modular curves, useful in the explicit computation of the intersection numbers obtained by pairing modular symbols to cusp forms. It is shown in [37] that the modular complex can be recovered canonically from the K-theory of the C^*-algebra (1.155).

Moreover, Mellin transforms of cusp forms of weight two for the congruence subgroups $\Gamma_0(p)$, with p prime, can be obtained by integrating along the boundary $\mathbb{P}^1(\mathbb{R})$ certain "automorphic series" defined in terms of the continued fraction expansion and of modular symbols.

These extensions of the theory of modular symbols to the noncommutative boundary appear to be interesting also in relation to the results of [10], where the pairing with modular symbols is used to give a formal analog of the Godbillon–Vey cocycle and to obtain a rational representative for the Euler class in the group cohomology $H^2(\mathrm{SL}_2(\mathbb{Q}), \mathbb{Q})$.

The fact that the arithmetic information on modular curves is stored in their noncommutative boundary (1.155) is interpreted in [38] as an instance of the physical principle of holography. Noncommutative spaces arising at the boundary of Shimura varieties have been further investigated by Paugam [41] from the point of view of Hodge structures.

This noncommutative boundary stratum of modular curves representing degenerations of lattices to pseudo-lattices has been proposed by Manin ([34] [35]) as a geometric space underlying the explicit class field theory problem for real quadratic fields. In fact, this is the first unsolved case of the Hilbert 12th problem. Manin developed in [34] a theory of real multiplication, where noncommutative tori and pseudolattices should play for real quadratic fields a role parallel to the one that lattices and elliptic curves play in the construction of generators of the maximal abelian extensions of imaginary quadratic fields. The picture that emerges from this "real multiplication program" is that the cases of $\mathbb{Q}$ (Kronecker–Weber) and of both imaginary and real quadratic fields should all have the same underlying geometry, related to different specializations of the GL_2 system. The relation of the GL_2 system and explicit class field theory for imaginary quadratic fields is analyzed in [13].

1.11 The BC algebra and optical coherence

It is very natural to look for concrete physical realizations of the phase transition exhibited by the BC system. An attempt in this direction has been proposed in [43], in the context of the physical phenomenon of quantum phase locking in lasers.

This interpretation relates the additive generators $e(r)$ of the BC algebra (*cf.* Proposition 1.6) with the quantum phase states, which are a standard tool in the theory of optical coherence (*cf. e.g.* [32]), but it leaves open the interpretation of the generators μ_n. Since on a finite dimensional Hilbert space isometries are automatically unitary, this rules out nontrivial representations of the μ_n in a fixed finite dimensional space.

After recalling the basic framework of phase states and optical coherence, we interpret the action of the μ_n as a "renormalization" procedure, relating the quantum phase states at different scales.

There is a well known analogy (*cf.* [46] §21-3) between the quantum statistical mechanics of systems with phase transitions, such as the ferromagnet or the Bose condensation of superconducting liquid Helium, and the physics of lasers, with the transition to single mode radiation being the analog of "condensation". The role of the inverse temperature β is played in laser physics by the "population inversion" parameter, with critical value at the inversion threshold. The injected signal of the laser acts like the external field responsible for the symmetry breaking mechanism. Given these identifications, one in fact obtains similar forms in the two systems for both thermodynamic potential and

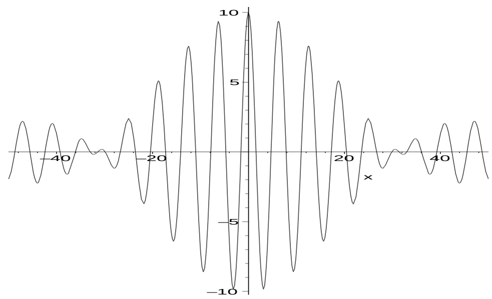

Figure 1.3: Output pulse train in lasers above threshold.

statistical distribution. The phase locking phenomenon is also analogous in systems with phase transitions and lasers, with the modes in the laser assuming same phase and amplitude above threshold being the analog of Cooper pairs of electrons acquiring the same energy and phase below critical temperature in superconductors.

In a laser cavity typically many longitudinal modes of the radiation are oscillating simultaneously. For a linewidth $\Delta\nu$ around a frequency ν_0 for the active medium in the cavity of length L and frequency spacing $\delta\nu = c/2L$, the number of oscillating modes is $N = \lceil \Delta\nu/\delta\nu \rceil$ and the field output of the laser is

$$E(x,t) = \sum_{n=-N/2}^{N/2} A_n \exp(-2\pi i \nu_n (t - x/c) + 2\pi i \theta_n), \tag{1.156}$$

with all the beat frequencies between adjacent modes $\nu_n - \nu_{n-1} = \delta\nu$. Due to noise in the cavity all these modes are uncorrelated, with a random distribution of amplitudes A_n and phases θ_n.

A mode locking phenomenon induced by the excited lasing atoms is responsible for the fact that, above the threshold of population inversion, the phases and amplitudes of the frequency modes become locked together. The resulting field

$$E(x,t) = Ae^{2\pi i\theta} \exp(-2\pi i \nu_0 (t - x/c)) \left(\frac{\sin(\pi\delta\nu(N+1)(t - x/c))}{\sin(\pi\delta\nu(t - x/c))} \right), \tag{1.157}$$

shows many locked modes behaving like a single longitudinal mode oscillating inside the cavity (*cf.* Figure 1.3). This phenomenon accounts for the typical narrowness of the laser linewidth and monocromaticity of laser radiation.

Since the interaction of radiation and matter in lasers is essentially a quantum mechanical phenomenon, the mode locking should be modeled by quantum mechanical phase operators corresponding to the resonant interaction of many different oscillators. In the quantum theory of radiation one usually describes a single mode by the Hilbert space spanned by the occupation number states $|n\rangle$,

with creation and annihilation operators a^* and a that raise and lower the occupation numbers and satisfy the relation $[a, a^*] = 1$. The polar decomposition $a = S\sqrt{N}$ of the annihilation operator is used to define a quantum mechanical phase operator, which is conjugate to the occupation number operator $N = a^* a$. Similar phase operators are used in the modeling of Cooper pairs. This approach to the definition of a quantum phase has the drawback that the one sided shift S is not a unitary operator. This can also be seen in the fact that the inverse Cayley transform of S, which gives the cotangent of the phase, is a symmetric non self-adjoint operator.

The emission of lasers above threshold can be described in terms of coherent state excitations,

$$|\alpha\rangle = \exp(-|\alpha|^2) \sum_n \frac{\alpha^n}{(n!)^{1/2}} |n\rangle,$$

which are eigenfunctions of the annihilation operators,

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle.$$

These are quantum mechanical analogs of classical electromagnetic waves as in (1.156) (1.157). One can show (*cf. e.g.* [32] §7.4) that the field excitation in a laser approaches a coherent state as the pumping increases to values above the population inversion threshold, with the phase diffusion governed by the equation of motion for quantum mechanical phase states.

The problem in defining a proper quantum phase operator, due to lack of self-adjointness, has been overcome by the following approximation of the basic quantum operators on the Fock space $\mathcal{H}$. One selects a scale, given by a positive integer $N \in \mathbb{N}$ and cuts down $\mathcal{H}$ to a finite dimensional subspace by the phase state projector

$$P_N = \sum_m |\theta_{m,N}\rangle\,\langle\theta_{m,N}|,$$

where the orthonormal vectors $|\theta_{m,N}\rangle$ in $\mathcal{H}$ are given by

$$|\theta_{m,N}\rangle := \frac{1}{(N+1)^{1/2}} \sum_{n=0}^{N} \exp\left(2\pi i \frac{m\,n}{N+1}\right) |n\rangle. \tag{1.158}$$

These are eigenvectors for the phase operators, that affect discrete values given by roots of unity, replacing a continuously varying phase.

This way, phase and occupation number behave like positions and momenta. An occupation number state has randomly distributed phase and, conversely, a phase state has a uniform distribution of occupation numbers.

We now realize the ground states of the BC system as representations of the algebra $\mathcal{A}$ in the Fock space $\mathcal{H}$ of the physical system described above. Given an embedding $\rho : \mathbb{Q}^{ab} \to \mathbb{C}$, which determines the choice of a ground state, the generators $e(r)$ and μ_n (*cf.* Proposition 1.6) act as

$$e(a/b)\,|n\rangle = \rho(\zeta^n_{a/b})\,|n\rangle,$$

$$\mu_k \, |n\rangle = |kn\rangle.$$

In the physical system, the choice of the ground state is determined by the primitive $N+1$-st root of unity

$$\rho(\zeta_{N+1}) = \exp(2\pi i/(N+1)).$$

One can then write (1.158) in the form

$$|\theta_{m,N}\rangle = e\left(\frac{m}{N+1}\right) \cdot v_N. \tag{1.159}$$

where we write v_N for the superposition of the first $N+1$ occupation states

$$v_N := \frac{1}{(N+1)^{1/2}} \sum_{n=0}^{N} |n\rangle.$$

Any choice of a primitive $N+1$-st root of unity would correspond to another ground state, and can be used to define analogous phase states. This construction of phase states brings in a new hidden group of symmetry, which is different from the standard rotation of the phase, and is the Galois group $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. This raises the question of whether such symmetries are an artifact of the approximation, or if they truly represent a property of the physical system.

In the BC algebra, the operators μ_n act on the algebra generated by the $e(r)$ by endomorphisms given by

$$\mu_n \, P(e(r_1), \dots, e(r_k)) \, \mu_n^* = \frac{\pi_n}{n^k} \sum_{ns=r} P(e(s_1), \dots, e(s_k)), \tag{1.160}$$

for an arbitrary polynomial P in k-variables, with $\pi_n = \mu_n \mu_n^*$ and $s = (s_1, \dots, s_k)$, $r = (r_1, \dots, r_k)$ in $(\mathbb{Q}/\mathbb{Z})^k$. In particular, this action has the effect of averaging over different choices of the primitive roots.

The averaging on the right hand side of (1.160), involving arbitrary phase observables $P(e(r_1), \dots, e(r_k))$, has physical meaning as statistical average over the choices of primitive roots. The left hand side implements this averaging as a renormalization group action.

Passing to the limit $N \to \infty$ for the phase states is a delicate process. It is known in the theory of optical coherence (*cf. e.g.* [33] §10) that one can take such limit only after expectation values have been calculated. The analogy between the laser and the ferromagnet suggests that this limiting procedure should be treated as a case of statistical limit, in the sense of [57]. In fact, when analyzing correlations near a phase transition, one needs a mechanism that handles changes of scale. In statistical mechanics, such mechanism exists in the form of a renormalization group, which expresses the fact that different length or energy scales are locally coupled. This is taken care here by the action of (1.160).

Bibliography

[1] J. Arledge, M. Laca, I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*, Doc. Math. 2 (1997) 115–138.

[2] M.W. Binder, *Induced factor representations of discrete groups and their types*, J. Functional Analysis, 115 (1993), 294–312.

[3] F. Boca and A. Zaharescu, *Factors of type III and the distribution of prime numbers*, Proc. London Math. Soc. (3) Vol.80 (2000), no. 1, 145–178.

[4] J.-B. Bost and A. Connes, *Produits Euleriens et facteurs de type III*, C. R. Acad. Sci. Paris Sér. I Math. Vol.315 (1992), no. 3, 279–284.

[5] J.B. Bost, A. Connes, *Hecke algebras, Type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (New Series) Vol.1 (1995) N.3, 411–457.

[6] O. Bratteli, D.W. Robinson. *Operator Algebras and Quantum Statistical Mechanics* I and II, Springer, New York 1979 and 1981.

[7] B. Brenken, *Hecke algebras and semigroup crossed product C^*-algebras*, Pacific J. Math., 187 (1999), 241–262.

[8] P.B. Cohen, *A C^*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*, Journal de Théorie des Nombres de Bordeaux 11 (1999) 15–30.

[9] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. (New Series) Vol.5 (1999) 29–106.

[10] A. Connes, H. Moscovici, *Modular Hecke Algebras and their Hopf Symmetry*, Moscow. Math. Journal, vol.4, n 1, (2004) 67–109.

[11] A. Connes, *A survey of foliations and operator algebras*, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., (1982) 521–628.

[12] A. Connes, M. Douglas, A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, J. High Energy Phys. 1998, no. 2, Paper 3, 35 pp. (electronic).

[13] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*, in preparation.

[14] J. Dixmier, *Les C^*-algèbres et leurs Représentations*, Gauthier-Villars, Paris 1964.

[15] H. Glöckner and G. Willis, *Topologisation of Hecke pairs and Hecke C^*-algebras*, (preprint), University of Newcastle, Australia (2002).

[16] R. Haag, *Local Quantum Physics*, Springer, Berlin 1992.

[17] R. Haag, N. M. Hugenholtz, M. Winnink *On the equilibrium states in quantum statistical mechanics*, Comm. Math. Phys. 5 (1967), 215–236.

[18] R. Hall, *Hecke C*-algebras*, PhD Thesis, Pennsylvania State University (1999).

[19] D. Harari, E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost–Connes au cas des corps globaux quelconques*, Selecta Math. (New Series) Vol.3 (1997) 205–243.

[20] B. Julia, *Statistical theory of numbers*, in "Number Theory and Physics", J.-M. Luck, P. Moussa and M. Waldschmidt (Eds.), Springer Verlag, Berlin, 1990.

[21] M. Laca, *Semigroups of $*$-endomorphisms, Dirichlet series, and phase transitions*, J. Funct. Anal. 152 (1998) 330–378.

[22] M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. (2) 61 (2000), no. 3, 893–904.

[23] M. Laca and N. Larsen, *Hecke algebras of semidirect products*, Proc. Amer. Math. Soc. (to appear in 2003).

[24] M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal., 139 (1996), 415–440.

[25] M. Laca and I. Raeburn, *A semigroup crossed product arising in number theory*. J. London Math. Soc. (2) 59 (1999), no. 1, 330–344.

[26] M. Laca and I. Raeburn, *The ideal structure of the Hecke C*-algebra of Bost and Connes*, Math. Ann. 318 (2000), no. 3, 433–451.

[27] M. Laca and M. van Frankenhuysen, *Phase transitions on Hecke C*-algebras and class field theory* (in preparation).

[28] S. Lang, *Elliptic Functions*, (Second Edition), Graduate Texts in Mathematics, Vol.112, Springer-Verlag 1987.

[29] N. Larsen, I. Putnam, I. Raeburn, *The two-prime analogue of the Hecke C^*-algebra of Bost and Connes*, Indiana Univ. Math. J. Vol.51 (2002), no. 1, 171–186.

[30] N. Larsen and I. Raeburn, *Representations of Hecke algebras and dilations of semigroup cross products*, J. London Math. Soc. (2) **66** (2002), no. 1, 198–212.

[31] E. Leichtnam and V. Nistor, *Cross product algebras and the homology of certain p-adic and adelic dynamical systems*, K-Theory, **21** (2000), 1–23.

[32] R. Loudon, *The Quantum Theory of Light*, third edition, Oxford University Press, 2000.

[33] L. Mandel, E. Wolf, *Optical Coherence and Quantum Optics*, Cambridge University Press, 1995.

[34] Yu.I. Manin, *Real Multiplication and noncommutative geometry*, preprint arXiv math.AG/0202109.

[35] Yu.I. Manin, *Von Zahlen und Figuren*, preprint arXiv math.AG/0201005.

[36] Yu.I. Manin, *Parabolic points and zeta functions of modular curves*, Math. USSR Izvestjia 6 (1972) N.1, 19–64.

[37] Yu.I. Manin, M. Marcolli, *Continued fractions, modular symbols, and non-commutative geometry*, Selecta Math. (New Series) Vol.8 (2002) N.3, 475–520.

[38] Yu.I. Manin, M. Marcolli, *Holography principle and arithmetic of algebraic curves*, Adv. Theor. Math. Phys. Vol.3 (2001) N.5, 617–650.

[39] D. Mumford, *Tata Lectures on Theta. I*, Progress in Mathematics, Vo.28, Birkhäuser, Boston, MA, 1983.

[40] S. Neshveyev, *Ergodicity of the action of the positive rationals on the group of finite adeles and the Bost-Connes phase transition theorem*, Proc. Amer. Math. Soc. 130 (2002), no. 10, 2999–3003.

[41] F. Paugam *Three examples of Noncommutative moduli spaces and related questions* Preprint 2004.

[42] G. Pedersen, *C^*-algebras and their Automorphism Groups*. Academic Press, New York 1979.

[43] M. Planat, *Invitation to the "spooky" quantum phase–locking effect and its link to $1/f$ fluctuations*, math-ph/0310082.

[44] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, NewYork 1987.

[45] D. Ruelle, *Statistical Mechanics*. World Scientific Publishing Co. Inc., River Edge, NJ, 1999. Reprint of the 1989 edition.

[46] M. Sargent, M. Scully, W. Lamb, *Laser Physics*, Addison–Wesley, 1974.

[47] G. Schlichting, *Polynomidentitäten und Permutationsdarstellungen lokalkompakter Gruppen*, Invent. Math. 55 (1979) N.2, 97–106.

[48] G. Schlichting, *Operationen mit periodischen Stabilisatoren*, Arch.Math. 34 (1980) N.2, 97–99.

[49] B. Schoeneberg, *Elliptic Modular Functions*, Springer-Verlag, New York - Heidelberg - Berlin, 1974.

[50] J.P. Serre, *Cours d'Arithmétique*, Presse Universitaire de France, 1977.

[51] G. Shimura, *Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton 1971.

[52] D. Spector, *Supersymmetry and the Möbius inversion function*, Commun. Math. Phys., Vol.127 (1990), 239–252.

[53] P. Stevenhagen, *Hilbert's 12th problem, complex multiplication and Shimura reciprocity*, Advanced Studies in Pure Math. 30 (2001) "Class Field Theory – its centenary and prospect" pp. 161–176.

[54] M. Takesaki, *Tomita's theory of modular Hilbert algebras and its applications.* Lecture Notes in Math., 28, Springer, 1970.

[55] K. Tzanev, *Hecke C*-algebras and amenability.* To appear in JOT.

[56] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Springer 1976.

[57] K.G. Wilson, *The renormalization group: critical phenomena and the Kondo problem*, Rev. Mod. Phys. Vol. 47 (1975) N.4, 773–840.

A. Connes
Collège de France, 3, rue Ulm, F-75005 Paris, France
I.H.E.S. 35 route de Chartres F-91440 Bures-sur-Yvette, France
Department of Mathematics, Vanderbilt University, TN-37240, USA
email: connes@ihes.fr, alain@connes.org

M. Marcolli
Max–Planck Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany
email: marcolli@mpim-bonn.mpg.de